EVT & Heavy Tails: Random measures and point processes; weak convergence to PRM and Lévy processes.

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1. Outline

1. Random measures and point processes; weak convergence to PRM and Lévy processes.
2. Multivariate regular variation; the Poisson transform; stable processes.
3. Some applications and inference examples using the point process method.
4. Some stylized applied probability network models: a renewal input model.

2. Introduction: Point processes and random measures.

2.1. Setup.

(Resnick, 2007, Chapter 3)
Setup for study of random measures, point processes and associated topology yielding *vague convergence*.

- $\mathbb{E}$: a nice space (lccb). Typically $\mathbb{E}$ is a finite dimensional Euclidean space: Subset of compactified $\mathbb{R}^d$ or $\mathbb{R}^d_+$.
- $\mathcal{E}$: Borel σ-algebra.
- Space of all Radon measures on $\mathbb{E}$:

  $$M_+(\mathbb{E}) = \{\mu : \mu \text{ is a non-negative measure on } \mathcal{E} \text{ and } \mu \text{ is Radon.}\} \tag{1}$$

  Note a measure μ is called *Radon*, if

  $$\mu(K) < \infty, \quad \forall K \in \mathcal{K}(\mathbb{E}) = \text{ compacta in } \mathbb{E}.$$

- Subset of $M_+(\mathbb{E})$ consisting of non-negative integer valued measures; ie, point measures: Call $m(\cdot) \in M_+(\mathbb{E})$ a point measure if

$$m(\cdot) = \sum_i \epsilon_{x_i}(\cdot), \quad x_i \in \mathbb{E}$$

where for $A \in \mathcal{E}$

$$\epsilon_{x_i}(A) = \begin{cases} 1, & \text{if } x_i \in A, \\ 0, & \text{if } x_i \notin A. \end{cases}$$

So $m(\cdot)$ is Radon: $m(K) < \infty$, for $K \in \mathcal{K}(\mathbb{E})$. Call $\{x_i\}$ the atoms or points and m is the function which counts how many atoms fall in a set. Then

$$M_p(\mathbb{E}) \subset M_+(\mathbb{E}) = \{ \text{ all Radon point measures.} \}$$

- Test functions: Those continuous functions which vanish on complements of compact sets:

$$C_K^+(\mathbb{E}) = \{f : \mathbb{E} \mapsto \mathbb{R}_+ : f \text{ is continuous with compact support.} \}$$

2.2. Convergence and topology in $M_+(\mathbb{E})$.

- Convergence concept: Vague convergence in $M_+(\mathbb{E})$ or $M_p(\mathbb{E})$: Sppse $\mu_n \in M_+(\mathbb{E})$, $n \geq 0$. Then $\mu_n \xrightarrow{v} \mu_0$, if for all $f \in C_K^+(\mathbb{E})$ we have

$$\mu_n(f) := \int_{\mathbb{E}} f(x)\mu_n(dx) \to \mu_0(f) := \int_{\mathbb{E}} f(x)\mu_0(dx), \quad (n \to \infty).$$

- Topology specified to make

$$\mu \mapsto \mu(f)$$

from

$$M_+(\mathbb{E}) \to \mathbb{R}_+$$

continuous for all $f \in C_K^+(\mathbb{E})$. Topology is metrizable as CSMS. Identify

$$\mu \leftrightarrow \{\mu(f), f \in C_K^+(\mathbb{E}),$$

or thinning $C_K^+(\mathbb{E})$ to a countable subset $\{f_i\}$

$$\mu \leftrightarrow \{\mu(f_i), i = 1, 2, \dots\} \subset \mathbb{R}_+^\infty.$$

- Leads to compactness criterion: A subset $M \subset M_+(\mathbb{E})$ vaguely relatively compact iff

$$\sup_{\mu \epsilon M} \mu(f) < \infty, \quad \forall f \in C_K^+(\mathbb{E}).$$

2.3. Random elements.

- The open subsets of $M_+(\mathbb{E})$ generate the Borel σ-field:

$$\mathcal{M}_+(\mathbb{E}) = \text{ Borel } \sigma\text{-field of subsets in } M_+(\mathbb{E}).\}$$

- Random element: So

$$\big(M_+(\mathbb{E}), \mathcal{M}_+(\mathbb{E})\big)$$

is measurable space. A random measure is measurable map

$$(\Omega, \mathcal{B}) \mapsto \big(M_+(\mathbb{E}), \mathcal{M}_+(\mathbb{E})\big)$$

and a point process is measurable map

$$(\Omega, \mathcal{B}) \mapsto \big(M_p(\mathbb{E}), \mathcal{M}_p(\mathbb{E})\big).$$

3. Weak convergence of random measures.

The following are usual and convenient methods of showing weak convergence of random elements in $M_+(\mathbb{E})$.

- In $M_+(\mathbb{E})$, random measures $\{\eta_n(\cdot), n \geq 0\}$ converge weakly

  $$\eta_n \Rightarrow \eta_0$$

  iff for any family $\{h_j\}$, with $h_j \in C_K^+(\mathbb{E})$ we have

  $$(\eta_n(h_j), j \geq 1) \Rightarrow (\eta_0(h_j), j \geq 1) \quad \text{in } \mathbb{R}^\infty.$$

  Nice:

  - One assumes a sequence $\{h_j\}$ and prove $\mathbb{R}^\infty$ convergence;
  - This reduces to proving $\mathbb{R}^d$-convergence for any fixed d.
  - Often, in fact, this reduced to one dimensional convergence.

- Method of Laplace functionals: A convenient transform technique for manipulating distributions of point processes and random measures.

Definition. Let

$$\eta : (\Omega, \mathcal{B}, \mathbb{P}) \mapsto (M_+(\mathbb{E}), \mathcal{M}_+(\mathbb{E}))$$

be a random measure. The *Laplace functional* of the random measure η is the non-negative function $\Psi_\eta : \{f : f : \mathbb{E} \mapsto \mathbb{R}_+\} \mapsto \mathbb{R}_+$

$$\Psi_\eta(f) = \mathbf{E}\exp\{-\eta(f)\} = \int_\Omega \exp\{-\eta(\omega, f)\} d\mathbb{P}(\omega)$$
$$= \int_{M_+(\mathbb{E})} \exp\{-\mu(f)\} \mathbb{P} \circ \eta^{-1}(d\mu).$$

Weak convergence of a sequence of random measures in $M_+(\mathbb{E})$ equivalent to the Laplace functionals of the random measures converging for each $f \in C_K^+(\mathbb{E})$. More detail: (Resnick, 1987, Section 3.5) or Kallenberg (1983), Neveu (1977).

Theorem 1 (Convergence criterion) *Let* $\{\eta_n, n \geq 0\}$ *be random elements of* $M_+(\mathbb{E})$*. Then*

$$\eta_n \Rightarrow \eta_0 \quad \textit{in } M_p(\mathbb{E}),$$

iff

$$\Psi_{\eta_n}(f) = \mathbf{E}e^{-\eta_n(f)} \to \mathbf{E}e^{-\eta_0(f)} = \Psi_{\eta_0}(f), \quad \forall f \in C_K^+(\mathbb{E}). \tag{2}$$

So weak convergence is characterized by convergence of Laplace functionals on $C_K^+(\mathbb{E})$.

$\frac{1}{2}$ **Proof.** Suppose $\eta_n \Rightarrow \eta_0$ in $M_+(\mathbb{E})$. The map $M_+(\mathbb{E}) \mapsto [0, \infty)$ defined by

$$\mu \mapsto \mu(f)$$

is continuous. The continuous mapping theorem gives

$$\eta_n(f) \Rightarrow \eta_0(f) \quad \text{in } \mathbb{R}.$$

Thus

$$e^{-\eta_n(f)} \Rightarrow e^{-\eta_0(f)},$$

and by Lebesgue's dominated convergence theorem

$$\mathbf{E}e^{-\eta_n(f)} \to \mathbf{E}e^{-\eta_0(f)}.$$

□

4. Poisson random measure

Let $N : (\Omega, \mathcal{B}) \mapsto (M_p(\mathbb{E}), \mathcal{M}_p(\mathbb{E}))$ be a point process with state space $\mathbb{E}$, where $\mathcal{M}_p(\mathbb{E})$ is the Borel σ-algebra of subsets of $M_p(\mathbb{E})$ generated by open sets.

Definition 1 *N is a Poisson process with mean measure μ or synonomously a Poisson random measure (PRM(μ)) if*

1. *For $A \in \mathcal{E}$*

$$P[N(A) = k] = \begin{cases} \dfrac{e^{-\mu(A)}(\mu(A))^k}{k!}, & \text{if } \mu(A) < \infty \\ 0, & \text{if } \mu(A) = \infty. \end{cases}$$

2. *If $A_1, \dots, A_k$ are disjoint subsets of $\mathbb{E}$ in $\mathcal{E}$, then $N(A_1), \dots, N(A_k)$ are independent random variables.*

4.1. Properties.

4.1.1. Transformation preserves Poisson-i-ness.

Proposition 1 *Suppose*

$$T : \mathbb{E} \mapsto \mathbb{E}'$$

is a measurable mapping of one nice space $\mathbb{E}$ *into another nice space* $\mathbb{E}'$ *such that*

$$K' \in \mathcal{K}(\mathbb{E}') \text{ is compact in } \mathbb{E}'$$
$$\Rightarrow$$
$$T^{-1}K' := \{e \in E : Te \in K'\} \in \mathcal{K}(\mathbb{E}).$$

If N *is* $\mathrm{PRM}(\mu)$ *on* $\mathbb{E}$ *then* $N' := N \circ T^{-1}$ *is* $\mathrm{PRM}(\mu')$ *on* $\mathbb{E}'$ *where* $\mu' := \mu \circ T^{-1}$.

Proof. Poisson marginals and independence preserved. □

4.1.2. Augmentation preserves Poisson-i-ness.

Proposition 2 *Suppose* $\{X_n\}$ *are random elements of a nice space* $\mathbb{E}_1$ *such that*

$$\sum_n \epsilon_{X_n}$$

is PRM*(*μ*). Suppose* $\{J_n\}$ *are iid random elements of a second nice space* $\mathbb{E}_2$ *with common probability distribution* F *and suppose the Poisson process and the sequence* $\{J_n\}$ *are defined on the same probability space and are independent. Then the point process on* $\mathbb{E}_1 \times \mathbb{E}_2$

$$\sum_n \epsilon_{(X_n, J_n)}$$

is PRM *with mean measure* $\mu \times F$*.*

4.1.3. Laplace functional of PRM(μ).

The Poisson process can be identified by the characteristic form of its Laplace functional.

Theorem 2 (Laplace functional of PRM) *The point process N is PRM(μ) iff its Laplace functional is of the form*

$$\Psi_N(f) = \exp\{-\int_{\mathbb{E}} (1 - e^{-f(x)})\mu(dx)\}, \quad f \in C_K^+(\mathbb{E}). \tag{3}$$

Proof. Verify (3) by *usual* measure theory argument:

- Let $f = \lambda 1_A$; for $\lambda > 0$. Then
$$\Psi_N(f) = E\exp\{-\lambda N(A)\}$$
and (3) verified by direct calculation.

- Let
$$f = \sum_{i=1}^{k} \lambda_i 1_{A_i}$$
for $\lambda_i > 0$ and $A_1, \dots, A_k$ a partition. Then again verify (3) by direct calculation.

- Finish with a limiting argument, approximating general f by simple $f_n \uparrow f$. □

4.1.4. Construction of PRM.

Suppose N is PRM(μ) on a nice space $\mathbb{E}$, such that

$$\mu(\mathbb{E}) < \infty.$$

Represent N as Poissonized sum of masses: Define the pm F

$$F(dx) = \mu(dx)/\mu(\mathbb{E})$$

on $\mathcal{E}$. Let

- $\{\boldsymbol{X}_n, n \geq 1\}$ iid random elements of $\mathbb{E}$, common distribution F;
- Let $\tau \perp\!\!\!\perp \{\boldsymbol{X}_n\}$, τ has Poisson distribution with parameter $\mu(\mathbb{E})$;
- Define
$$N = \begin{cases} \sum_{i=1}^{\tau} \epsilon_{\boldsymbol{X}_i}, & \text{if } \tau \geq 1 \\ 0, & \text{if } \tau = 0. \end{cases}$$

Then N is PRM(μ).

Proof. Compute the Laplace functional. □

5. Sample measures and PRM

Suppose that for each $n \geq 1$ we have

$$\{\boldsymbol{X}_{n,j}, j \geq 1\}$$

is a sequence of iid random elements of $(\mathbb{E}, \mathcal{E})$. We call

$$\sum_{j=1}^{n} \epsilon_{\boldsymbol{X}_{n,j}}$$

a *sample measure*.

Theorem 3 (Basic convergence) *We have*

$$\sum_{j=1}^{n} \epsilon_{\boldsymbol{X}_{n,j}} \Rightarrow \sum_{k} \epsilon_{\boldsymbol{j}_k} = \mathrm{PRM}(\mu) \tag{4}$$

in $M_p(\mathbb{E})$, *iff*

$$n\mathbb{P}[\boldsymbol{X}_{n,1} \in \cdot] = \mathbf{E}\left(\sum_{j=1}^{n} \epsilon_{\boldsymbol{X}_{n,j}}(\cdot)\right) \xrightarrow{v} \mu \tag{5}$$

in $M_+(\mathbb{E})$. *Furthermore* (4) *or* (5) *is equivalent to the version with a "time" component:*

$$\sum_{j=1}^{n} \epsilon_{\left(\frac{j}{n}, \boldsymbol{X}_{n,j}\right)} \Rightarrow \sum_{k} \epsilon_{\left(t_k, \boldsymbol{j}_k\right)} = \mathrm{PRM}(\mathbb{LEB} \times \mu) \tag{6}$$

in $M_p([0,\infty) \times \mathbb{E})$.

Proof. Compute Laplace functionals of the sample measures and decide when they converge: For $f \in C_K^+(\mathbb{E})$,

$$\begin{aligned}\mathbf{E}e^{-\sum_{j=1}^n \epsilon_{\boldsymbol{X}_{n,j}}(f)} =& \mathbf{E}e^{-\sum_{j=1}^n f(\boldsymbol{X}_{n,j})} = \left(\mathbf{E}e^{-f(\boldsymbol{X}_{n,1})}\right)^n \\ =& \left(1 - \frac{\mathbf{E}(n(1-e^{-f(\boldsymbol{X}_{n,1})}))}{n}\right)^n \\ =& \left(1 - \frac{\int_{\mathbb{E}}(1-e^{-f(x)})nP[\boldsymbol{X}_{n,1} \in dx]}{n}\right)^n\end{aligned}$$

and this converges to

$$\exp\{\int_{\mathbb{E}}(1-e^{-f(x)})\mu(dx)\},$$

the Laplace functional of PRM(μ), iff

$$\int_{\mathbb{E}}(1-e^{-f(x)})nP[\boldsymbol{X}_{n,1} \in dx] \to \int_{\mathbb{E}}(1-e^{-f(x)})\mu(dx).$$

and this last statement is equivalent to vague convergence in (5). □

Corollary 1 (Variant of basic convergence) *Suppose additionally that* $0 < a_n \uparrow \infty$*. Then for a measure* $\mu \in M_+(\mathbb{E})$ *we have*

$$\frac{1}{a_n}\sum_{j=1}^{n} \epsilon_{X_{n,j}} \Rightarrow \mu \tag{7}$$

on $M_+(\mathbb{E})$ *iff*

$$\frac{n}{a_n} P[X_{n,1} \in \cdot\,] = \mathbf{E}\left(\frac{1}{a_n}\sum_{j=1}^{n} \epsilon_{X_{n,j}}(\cdot)\right) \xrightarrow{v} \mu \tag{8}$$

in $M_+(\mathbb{E})$.

Proof. Similar to previous proof. Compute Laplace functional for the quantity on the left side of (7):

$$\mathbf{E}e^{-\frac{1}{a_n}\sum_{i=1}^{n}\epsilon_{X_{n,1}}(f)} = \left(\mathbf{E}e^{-\frac{1}{a_n}f(X_{n,1})}\right)^n$$
$$= \left(1 - \frac{\int_{\mathbb{E}}\left(1-e^{-\frac{1}{a_n}f(x)}\right)n\mathbb{P}[X_{n,1}\in dx]}{n}\right)^n$$

and this converges to $e^{-\mu(f)}$, the Laplace functional of μ, iff

$$\int_{\mathbb{E}}(1-e^{-\frac{1}{a_n}f(x)})nP[X_{n,1}\in dx] \to \mu(f). \tag{9}$$

Since $a_n \to \infty$,

$$\int_{\mathbb{E}}(1-e^{-f(x)/a_n})n\mathbb{P}[X_{n,1}\in dx] \approx \int_{\mathbb{E}} f(x)\frac{n}{a_n}P[X_{n,1}\in dx] \to \mu(f)$$

iff

$$\frac{n}{a_n}P[X_{n,1}\in\cdot\,] \xrightarrow{v} \mu.$$

□

6. Regular variation and measures; review

6.1. $d = 1$, $\mathbb{CONE} = (0, \infty]$.

Suppose $Z \geq 0$ is a random variable with distribution function F. Regular variation of $1 - F$ has the following equivalences:

(i) $\bar{F} \in RV_{-\alpha}$, $\alpha > 0$.

(ii) There exists a sequence $\{b_n\}$, with $b_n \to \infty$ such that

$$\lim_{n\to\infty} n\bar{F}(b_n x) = x^{-\alpha}, \quad x > 0.$$

(iii) There exists a sequence $\{b_n\}$ with $b_n \to \infty$ such that

$$\mu_n(\cdot) := n\mathbb{P}\left[\frac{Z}{b_n} \in \cdot\right] \xrightarrow{v} \nu_\alpha(\cdot) \tag{10}$$

in $M_+(0, \infty]$, where $\nu_\alpha(x, \infty] = x^{-\alpha}$.

6.2. $d > 1$, $\mathbb{CONE} = \mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

Equivalences for regular variation of probability measures on $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

Theorem 4 1. *There exists a Radon measure* ν *on* $\mathbb{E}$ *such that*

$$\lim_{t\to\infty} \frac{1-F(t\mathbf{x})}{1-F(t\mathbf{1})} = \lim_{t\to\infty} \frac{\mathbb{P}\big[\frac{\boldsymbol{Z}}{t} \in [\mathbf{0},\mathbf{x}]^c\big]}{\mathbb{P}\big[\frac{\boldsymbol{Z}}{t} \in [\mathbf{0},\mathbf{1}]^c\big]} = \nu\big([\mathbf{0},\mathbf{x}]^c\big), \qquad (11)$$

for all points $\mathbf{x} \in [\mathbf{0},\boldsymbol{\infty}) \setminus \{\mathbf{0}\}$ *which are continuity points of the function* $\nu([\mathbf{0},\cdot]^c)$.

2. *There exists a function* $b(t) \to \infty$ *and a Radon measure* ν *on* $\mathbb{E}$ *called the limit measure, such that in* $M_+(\mathbb{E})$

$$t\mathbb{P}\Big[\frac{\boldsymbol{Z}}{b(t)} \in \cdot\Big] \xrightarrow{v} \nu, \quad t \to \infty. \qquad (12)$$

3. *There exists a sequence* $b_n \to \infty$ *and a Radon measure* ν *on* $\mathbb{E}$ *such that in* $M_+(\mathbb{E})$

$$n\mathbb{P}\Big[\frac{\boldsymbol{Z}}{b_n} \in \cdot\Big] \xrightarrow{v} \nu, \quad n \to \infty. \qquad (13)$$

4. *Polar coordinate version: Define*

$$(R, \boldsymbol{\Theta}) = (\|\boldsymbol{Z}\|, \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|}$$

and

$$\aleph_+ = \{\mathbf{x} \in \mathbb{E} : \|\mathbf{x}\| = 1\}.$$

There exists a probability measure $S(\cdot)$ *on* $\aleph_+$ *called the angular measure, and a function* $b(t) \to \infty$ *such that for* $(R, \boldsymbol{\Theta}) = \left(\|\boldsymbol{Z}\|, \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|}\right)$ *we have*

$$t\mathbb{P}[(\frac{R}{b(t)}, \boldsymbol{\Theta}) \in \cdot] \xrightarrow{v} c\nu_\alpha \times S \tag{14}$$

in $M_+\big((0,\infty] \times \aleph_+\big)$ *, for some* $c > 0$.

5. *There exists a probability measure* $S(\cdot)$ *on* $\aleph_+$ *and a sequence* $b_n \to \infty$ *such that for* $(R, \boldsymbol{\Theta}) = \left(\|\boldsymbol{Z}\|, \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|}\right)$ *we have*

$$n\mathbb{P}[(\frac{R}{b_n}, \boldsymbol{\Theta}) \in \cdot] \xrightarrow{v} c\nu_\alpha \times S \tag{15}$$

in $M_+\big((0,\infty] \times \aleph_+\big)$, *for some* $c > 0$.

7. Regular variation and the Poisson random measure

Multivariate regular variation additionally equivalent to induced sample measures weakly converging to Poisson random measure limits.

Theorem 5 *Suppose* $\{\boldsymbol{Z}, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \dots\}$ *are iid; after transformation with polar coordinates the sequence is* $\{(R, \boldsymbol{\Theta}), (R_1, \boldsymbol{\Theta}_1), (R_2, \boldsymbol{\Theta}_2), \dots\}$. *Any of the equivalences in Theorem 4, are also equivalent to*

6. *There exists* $b_n \to \infty$ *such that*

$$\sum_{i=1}^{n} \epsilon_{\boldsymbol{Z}_i/b_n} \Rightarrow \mathrm{PRM}(\nu), \tag{16}$$

in $M_p(\mathbb{E})$.

7. *There exists a sequence* $b_n \to \infty$ *such that*

$$\sum_{i=1}^{n} \epsilon_{(R_i/b_n, \Theta_i)} \Rightarrow \mathrm{PRM}(c\nu_\alpha \times S) \tag{17}$$

in $M_p((0, \infty] \times \aleph_+)$.

These conditions imply that for any sequence $k = k(n) \to \infty$ *such that* $n/k \to \infty$ *we have*

8. *In* $M_+(\mathbb{E})$

$$\frac{1}{k}\sum_{i=1}^{n}\epsilon_{\boldsymbol{Z}_i/b(\frac{n}{k})} \Rightarrow \nu \tag{18}$$

and

9. *In* $M_+\big((0,\infty] \times \aleph_+\big)$

$$\frac{1}{k}\sum_{i=1}^{n}\epsilon_{(R_i/b(\frac{n}{k}),\boldsymbol{\Theta}_i)} \Rightarrow c\nu_\alpha \times S \tag{19}$$

and 8 or 9 is equivalent to any of 1–7, provided $k(\cdot)$ *satisfies* $k(n) \sim k(n+1)$.

7.1. Variant with time coordinate

The result with a time coordinate needed for proving weak convergence of partial sum processes or maximal processes in the space $D[0,\infty)$.

Theorem 6 *Suppose* $\{\boldsymbol{Z}, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \dots\}$ *are iid random elements of* $[\mathbf{0}, \boldsymbol{\infty})$. *Then multivariate regular variation of the distribution of* $\boldsymbol{Z}$ *in* $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$

$$n\mathbb{P}[\frac{\boldsymbol{Z}}{b_n} \in \cdot] \xrightarrow{v} \nu$$

is also equivalent to

$$\sum_j \epsilon_{(\frac{j}{n}, \boldsymbol{Z}_j/b_n)} \Rightarrow \text{PRM}(\mathbb{LEB} \times \nu) \tag{20}$$

in $M_+([0,\infty) \times \mathbb{E})$.

8. Examples: How to use.

8.1. Weak Convergence to Extremal processes.

Suppose $\{\boldsymbol{X}_j, j \geq 1\}$ are iid in $\mathbb{R}^d$ with common distn $F \in MDA(G)$. Then DOA $\Rightarrow$

$$nP\Big[\frac{\boldsymbol{X}_1 - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \cdot\Big] \xrightarrow{v} \nu(\cdot)$$

where

$$G(\mathbf{x}) = \exp\{-\nu(-\boldsymbol{\infty}, \mathbf{x}]^c\}.$$

Suppose for specificity that G has Gumbel marginals. Then

$$\sum_i \epsilon_{\big(i/n, (\boldsymbol{X}_i - \boldsymbol{b}(n))/\boldsymbol{a}(n)\big)} \Rightarrow \sum_i \epsilon_{(t_k, \boldsymbol{j}_k)}(\cdot) = PRM(\mathbb{LEB} \times \nu), \qquad (21)$$

in $M_p([-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{\boldsymbol{\infty}\})$. Consider the functional

$$M_p([-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{\boldsymbol{\infty}\}) \to D(0, \infty)$$

defined by

$$\sum_i \epsilon_{(u_k, v_k)} \mapsto x(t) = \bigvee_{u_k \leq t} v_k,$$

and apply it to (21)

$$\bigvee_{i/n \leq t} (\boldsymbol{X}_i - \boldsymbol{b}(n))/\boldsymbol{a}(n) \Rightarrow \boldsymbol{Y}(t) = \bigvee_{t_k \leq t} \boldsymbol{j}_k, \quad \text{in } D(0, \infty).$$

8.2. How to estimate the angular measure.

Suppose we have converted regular variation to standard form (either by the rank method, the powering up method or some other method). Or perhaps the limit relation is not quite standard and is in the form:

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{\boldsymbol{Z}_i/b(\frac{n}{k})} \Rightarrow \nu_*, \qquad \text{in } M_+(\mathbb{E}), \tag{22}$$

$$\frac{1}{k}\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)} \Rightarrow (c\nu_\alpha \times S), \qquad \text{in } M_+((0,\infty] \times \aleph), \tag{23}$$

provided $k = k(n) \to \infty$ and $k/n \to 0$. Except for the fact that we don't know $b(\cdot)$, the LHS of 22 is a consistent estimator of ν_*. Similarly, a consistent estimator (if we knew $b(\cdot)$) of S is

$$\frac{\sum_{i=1}^{n} \epsilon_{(R_i/b(n/k),\Theta_i)}[1,\infty] \times \cdot)}{\sum_{i=1}^{n} \epsilon_{R_i/b(n/k)}[1,\infty]}.$$

This is the relative frequency of those Θ's associated with big R's where *big* means bigger than $b(n/k)$.

8.3. Why the rank transform preserves asymptotic information about the angular measure.

Assume $d = 1$; the argument is virtually the same for $d > 1$. Start with

$$tP[\frac{X_1}{b(t)} \in \cdot] \to \nu_\alpha.$$

Then the non-decreasing processes converge:

$$\xi_n(t) := \frac{1}{k}\sum_{i=1}^{n} \epsilon_{X_i/b(\frac{n}{k})}(t^{-1},\infty] \Rightarrow \nu_\alpha(t^{-1},\infty] = t^\alpha = \xi_\infty(t), \qquad (24)$$

in $D[0,\infty)$. So inverses converge:

$$\xi_n^{\leftarrow} = \left(\frac{X_{([kt])}}{b(\frac{n}{k})}\right)^{-1} \Rightarrow t^{1/\alpha}, \qquad \text{in } D[0,\infty),$$

where $X_{(j)}$ is today's notation for the jth largest order statistic. This means

$$\frac{X_{([ks])}}{b(\frac{n}{k})} \Rightarrow s^{-1/\alpha}, \qquad \text{in } D(0,\infty]. \qquad (25)$$

Couple (24) and (25) into joint convergence:

$$\left(\frac{1}{k}\sum_{i=1}^{n} \epsilon_{X_i/b(\frac{n}{k})}(\cdot), \frac{X_{([k\cdot])}}{b(\frac{n}{k})}\right) \Rightarrow \left(\nu_\alpha(\cdot), (\cdot)^{-1/\alpha}\right)$$

and apply composition

$$\Big(\frac{1}{k}\sum_{i=1}^{n}\epsilon_{X_i/b(\frac{n}{k})}\big(\frac{X_{([k\cdot])}}{b(\frac{n}{k})},\infty\big] \Rightarrow \nu_\alpha(t^{-1/\alpha},\infty],$$

or

$$\frac{1}{k}\sum_{i=1}^{n}1_{[X_i\geq X_{([kt])}]} \Rightarrow \nu_\alpha(t^{-1/\alpha},\infty].$$

Now

$$X_i \geq [kt]th \text{ largest order statistics iff } r_i \leq kt,$$

and therefore

$$\frac{1}{k}\sum_{i=1}^{n}1_{[r_i/k\leq t]} \Rightarrow \nu_\alpha(t^{-1/\alpha},\infty].$$

Set $s = 1/t$ to get

$$\frac{1}{k}\sum_{i=1}^{n}1_{[r_i/k\leq s^{-1}]} \Rightarrow \nu_\alpha(s^{1/\alpha},\infty],$$

or

$$\frac{1}{k}\sum_{i=1}^{n}\epsilon_{k/r_i}(s,\infty] \Rightarrow \nu_\alpha(s^{1/\alpha},\infty] = s^{-1}.$$

In general, the vector version of this argument goes the same and we get

$$\frac{1}{k}\sum_{i=1}^{n}\epsilon_{k/\boldsymbol{r}_i} \Rightarrow \nu_*$$

where

$$\nu_*(c\cdot) = c^{-1}\nu_*(\cdot).$$

8.4. Weak convergence of partial sums to Lévy processes

Result for d dimensions. Set $\mathbb{E} = [-\infty, \infty] \setminus \{\mathbf{0}\}$. Denote random vectors by $\boldsymbol{X} = (X^{(1)}, \ldots, X^{(d)})$.

Theorem 7 *Suppose for each $n \geq 1$, that $\{\boldsymbol{X}_{n,j}, j \geq 1\}$ are iid random vectors such that*

$$n\mathbb{P}[\boldsymbol{X}_{n,1} \in \cdot] \xrightarrow{v} \nu(\cdot) \tag{26}$$

in $M_+(\mathbb{E})$, where ν is a Lévy measure and for each $j = 1, \ldots, d$,

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} n\mathbf{E}\Big(\big(X_{n,1}^{(j)}\big)^2 1_{[|X_{n,1}^{(j)}| \leq \varepsilon]}\Big) = 0. \tag{27}$$

Define the partial sum process based on the nth array row:

$$\boldsymbol{X}_n(t) := \sum_{k=1}^{[nt]} \Big(\boldsymbol{X}_{nk} - \mathbf{E}\big(\boldsymbol{X}_{n,k} 1_{[\|\boldsymbol{X}_{n,k}\| \leq 1]}\big)\Big), \quad t \geq 0.$$

Then (26) and (27) imply

$$\boldsymbol{X}_n \Rightarrow \boldsymbol{X}_0,$$

in $D([0,\infty), \mathbb{R}^d)$, where $\boldsymbol{X}_0(\cdot)$ is a Lévy jump process with Lévy measure ν.

8.4.1. Special case. Weak convergence of partial sums of iid regularly varying rv's to stable processes

We can specialize the result for convergence of sums to Lévy processes. Assume, for simplicity, $d = 1$ and

$$\mathbb{CONE} = [-\infty, \infty] \setminus \{0\}.$$

Regular variation of the tail probabilities on the cone $\mathbb{R} \setminus \{0\}$,

$$t\mathbb{P}[\frac{Z_1}{b(t)} \in \cdot\,] \stackrel{v}{\to} \nu(\cdot),$$

is equivalent to

$$\lim_{x\to\infty} \frac{\mathbb{P}[Z_1 > x]}{\mathbb{P}[|Z_1| > x]} = px^{-\alpha}, \quad \lim_{x\to\infty} \frac{\mathbb{P}[Z_1 \leq -x]}{\mathbb{P}[|Z_1| > x]} = qx^{-\alpha},$$

and

$$\mathbb{P}[|Z_1| > x] \in RV_{-\alpha}.$$

Corollary 2 *Consider the special case where* $\{Z_n, n \geq 1\}$ *are iid random variables on* $\mathbb{R}$ *and set* $X_{n,j} = Z_j/b_n$ *for some* $b_n \to \infty$. *Define* ν *for* $x > 0$ *and* $0 < \alpha < 2$ *by*

$$\nu((x, \infty]) = px^{-\alpha}, \quad \nu((-\infty, -x]) = qx^{-\alpha}, \tag{28}$$

where $0 \leq p \leq 1$ *and* $q = 1 - p$. *Then*

$$\sum_{j=1}^{[n\cdot]} \frac{Z_j}{b(n)} - [n\cdot]\mathbf{E}\Big(\frac{Z_1}{b(n)} 1_{[|\frac{Z_1}{b(n)}|\leq 1]}\Big) \Rightarrow X_\alpha(\cdot), \tag{29}$$

in $D[0, \infty)$, *where the limit is* α*-stable Lévy motion with Lévy measure* ν, *iff*

$$n\mathbb{P}[\frac{Z_1}{b_n} \in \cdot] \xrightarrow{v} \nu, \tag{30}$$

in $M_+(\mathbb{CONE})$.

Proof sufficiency for the special one-dimensional case. Given regular variation, plug into the result about convergence of sums to a Lévy process. We only need to check the *truncated 2nd moment condition*

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} n\mathbf{E}\Big(\big(X_{n,1}\big)^2 1_{[\,|X_{n,1}|\le\varepsilon]}\Big)=0,$$

where

$$X_{n,1}=\frac{Z_1}{b(n)}.$$

We have by Karamata's theorem,

$$n\mathbf{E}\left(\Big(\frac{Z_1}{b(n)}\Big)^2 1_{\left[\,|\frac{Z_1}{b(n)}|\le\varepsilon\right]}\right)\to\int_{[\,|x|\le\varepsilon\,]} x^2\nu(dx),\quad (n\to\infty)$$

$$=\frac{p\alpha\varepsilon^{2-\alpha}}{2-\alpha}+\frac{q\alpha\varepsilon^{2-\alpha}}{2-\alpha}=(\text{ const })\varepsilon^{2-\alpha}$$

and as $\varepsilon\to 0$, we have $\varepsilon^{2-\alpha}\to 0$ as required for the partial sum process to converge to the Lévy process.

Proof of general result from Resnick and Greenwood (1979), Resnick (2007). Proof in several steps:

STEP 1: Basic convergence, Theorem 3, and (26) imply

$$\sum_{k=1}^{\infty} \epsilon_{(\frac{k}{n}, \boldsymbol{X}_{n,k})} \Rightarrow \sum_{k} \epsilon_{(t_k, \boldsymbol{j}_k)} = \text{PRM}(\mathbb{LEB} \times \nu) \tag{31}$$

in $M_p([0,\infty) \times \mathbb{E})$.

STEP 2: Two continuity assertions:

(i) With respect to the distribution of PRM $(\mathbb{LEB} \times \nu)$, the restriction map $M_p([0,\infty) \times \mathbb{E}) \mapsto M_p([0,\infty) \times \{\mathbf{x} : \|\mathbf{x}\| > \varepsilon\})$ defined by

$$m \mapsto m|_{[0,\infty) \times \{\mathbf{x}:\|\mathbf{x}\|>\varepsilon\}}$$

is almost surely continuous.

(ii) On $M_p([0,\infty] \times \{\mathbf{x} : \|\mathbf{x}\| > \varepsilon\})$ the summation functional from $M_p([0,\infty) \times \{\mathbf{x} : \|\mathbf{x}\| > \varepsilon\}) \mapsto D([0,T], \mathbb{R}^d)$ defined by

$$\sum_{k} \epsilon_{(\tau_k, \boldsymbol{J}_k)} \to \sum_{\tau_k \le (\cdot)} \boldsymbol{J}_k$$

is almost surely continuous with respect to the distribution of $\text{PRM}(\mathbb{LEB} \times \nu)$.

STEP 3: From first continuity assertion in Step 2, the convergence statement in Step 1, and continuous mapping Theorem we get the restricted convergence

$$\sum_k 1_{[\|\boldsymbol{X}_{n,k}\|>\varepsilon]}\epsilon_{(\frac{k}{n},\boldsymbol{X}_{n,k})} \Rightarrow \sum_k 1_{[\|\boldsymbol{j}_k\|>\varepsilon]}\epsilon_{(t_k,\boldsymbol{j}_k)} \tag{32}$$

in $M_p([0,\infty)\times\{\mathbf{x}:\|\mathbf{x}\|>\varepsilon\})$. From the second continuity assertion in Step 2, we get from (32)

$$\sum_{k=1}^{[n\cdot]} \boldsymbol{X}_{n,k} 1_{[\|\boldsymbol{X}_{n,k}\|>\varepsilon]} \Rightarrow \sum_{t_k\le(\cdot)} \boldsymbol{j}_k 1_{[\|\boldsymbol{j}_k\|>\varepsilon]} \tag{33}$$

in $D([0,T],\mathbb{R}^d)$. Similarly we get

$$\sum_{k=1}^{[n\cdot]} \boldsymbol{X}_{n,k} 1_{[\varepsilon<\|\boldsymbol{X}_{n,k}\|\le 1]} \Rightarrow \sum_{t_k\le(\cdot)} \boldsymbol{j}_k 1_{[\varepsilon<\|\boldsymbol{j}_k\|\le 1]}. \tag{34}$$

STEP 4. In (34), take expectations and apply (26) to get

$$[n\cdot]\mathbf{E}\big(\boldsymbol{X}_{n,1}1_{[\varepsilon<\|\boldsymbol{X}_{n,1}\|\le 1]}\big) \to (\cdot)\int_{\{\mathbf{x}:\varepsilon<\|\mathbf{x}\|\le 1\}} \mathbf{x}\nu(d\mathbf{x}) \tag{35}$$

in $D([0,T],\mathbb{R}^d)$. To justify this, observe first for any $t>0$ that

$$[nt]\mathbf{E}\big(\boldsymbol{X}_{n,1}1_{[\varepsilon<\|\boldsymbol{X}_{n,1}\|\le 1]}\big) = \frac{[nt]}{n}\int_{\{\mathbf{x}:\|\mathbf{x}\|\in(\varepsilon,1]\}} \mathbf{x}n\mathbb{P}[\boldsymbol{X}_{n,1}\in d\mathbf{x}]$$
$$\to t\int_{\{\mathbf{x}:\|\mathbf{x}\|\in(\varepsilon,1]\}} \mathbf{x}\nu(d\mathbf{x})$$

since $n\mathbb{P}[\boldsymbol{X}_{n,1}\in\cdot] \xrightarrow{v} \nu(\cdot)$. Convergence is locally uniform in t and hence convergence takes place in $D([0,T],\mathbb{R}^d)$.

STEP 5. Difference (33)–(35). The result is

$$\boldsymbol{X}_n^{(\varepsilon)}(\cdot) = \sum_{k=1}^{[n\cdot]} \boldsymbol{X}_{n,k}1_{[\,\|\boldsymbol{X}_{n,k}\|>\varepsilon]} - [n\cdot]\mathbf{E}\big(\boldsymbol{X}_{n,1}1_{[\varepsilon<\|\boldsymbol{X}_{n,1}\|\le 1]}\big)$$
$$\Rightarrow \boldsymbol{X}_0^{(\varepsilon)}(\cdot) := \sum_{t_k\le\cdot} \boldsymbol{j}_k 1_{[\,\|\boldsymbol{j}_k\|>\varepsilon]} - (\cdot)\int_{\left\{\mathbf{x}:\,\|\mathbf{x}\|\in(\varepsilon,1]\right\}} \mathbf{x}\nu(d\mathbf{x}). \tag{36}$$

From the Itô representation of a Lévy process, for almost all ω, as $\varepsilon\downarrow 0$,

$$\boldsymbol{X}_0^{(\varepsilon)}(\cdot)\to\boldsymbol{X}_0(\cdot),$$

locally uniformly in t. Let $d(\cdot,\cdot)$ be the Skorohod metric on $D[0,\infty)$. Local uniform convergence $\Rightarrow$ implies Skorohod convergence, so

$$d\left(\boldsymbol{X}_0^{(\varepsilon)}(\cdot),\boldsymbol{X}_0(\cdot)\right)\to 0$$

almost surely as $\varepsilon \downarrow 0$. Almost sure convergence $\Rightarrow$ weak convergence, so

$$\boldsymbol{X}_0^{(\varepsilon)}(\cdot) \Rightarrow \boldsymbol{X}_0(\cdot).$$

in $D([0,\infty),\mathbb{R}^d)$.

STEP 6. By Converging Together Theorem suffices to show

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}[d(\boldsymbol{X}_n^{(\varepsilon)},\boldsymbol{X}_n)>\delta]=0.$$

Convergence in $D([0,\infty),\mathbb{R}^d)$, if Skorohod convergence in $D([0,T],\mathbb{R}^d)$ for any T. Skorohod metric on $D([0,T],\mathbb{R}^d)$ bounded above by the uniform metric on $D([0,T],\mathbb{R}^d)$. Suffices to show

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\mathbb{P}[\sup_{0\le t\le T}\|\boldsymbol{X}_n^{(\varepsilon)}(t)-\boldsymbol{X}_n(t)\|>\delta]=0,$$

for any $\delta>0$. Recalling definitions of $\boldsymbol{X}_n^{(\epsilon)}$ and $\boldsymbol{X}_n$:

$$\begin{aligned}\|\boldsymbol{X}_n^{(\varepsilon)}(t)-\boldsymbol{X}_n(t)\| =&\Big\|\sum_{k=1}^{[nt]}\boldsymbol{X}_{n,k}1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}-[nt]\mathbf{E}\big(\boldsymbol{X}_{n,1}1_{[\,\|\boldsymbol{X}_{n,1}\|\le\varepsilon]}\big)\Big\|\\ =&\Big\|\sum_{k=1}^{[nt]}\Big(\boldsymbol{X}_{n,k}1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}-\mathbf{E}\big(\boldsymbol{X}_{n,k}1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}\big)\Big)\Big\|\end{aligned}$$

and so

$$\mathbb{P}[\sup_{0\le t\le T} \|\boldsymbol{X}_n^{(\varepsilon)}(t) - \boldsymbol{X}_n(t)\| > \delta]$$

$$\le \mathbb{P}\Big[\sup_{0\le t\le T} \Big\|\sum_{k=1}^{[nt]} \Big(\boldsymbol{X}_{n,k} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]} - \mathbf{E}\big(\boldsymbol{X}_{n,k} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}\big)\Big)\Big\| > \delta\Big]$$

$$= \mathbb{P}\Big[\sup_{0\le j\le nT} \Big\|\sum_{k=1}^{j} \Big(\boldsymbol{X}_{n,k} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]} - \mathbf{E}\big(\boldsymbol{X}_{n,k} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}\big)\Big)\Big\| > \delta\Big].$$

Now use the fact that $\|\mathbf{x}\| \le d \bigvee_{i=1}^{d} |x^{(i)}|$ and we get the bound

$$\le \sum_{i=1}^{d} \mathbb{P}\Big[\sup_{0\le j\le nT} \Big|\sum_{k=1}^{j} \Big(X_{n,k}^{(i)} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]} - \mathbf{E}\big(X_{n,k}^{(i)} 1_{[\,\|\boldsymbol{X}_{n,k}\|\le\varepsilon]}\big)\Big)\Big| > \frac{\delta}{d}\Big]$$

and by Kolmogorov's inequality this has upper bound

$$\begin{aligned}
&\leq (\delta/d)^{-2} \sum_{i=1}^{d} \mathrm{Var}\Big(\sum_{k=1}^{[nT]} X_{n,k}^{(i)} 1_{[\,\|\boldsymbol{X}_{n,k}\| \leq \varepsilon]}\Big) \\
&= (\delta/d)^{-2} \sum_{i=1}^{d} [nT] \mathrm{Var}\Big(X_{n,1}^{(i)} 1_{[\,\|\boldsymbol{X}_{n,1}\| \leq \varepsilon]}\Big) \\
&\leq (\delta/d)^{-2} \sum_{i=1}^{d} [nT] \mathbf{E}\Big(\big(X_{n,1}^{(i)}\big)^2 1_{[\,|X_{n,1}^{(i)}| \leq \varepsilon]}\Big).
\end{aligned}$$

Taking $\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty}$, we easily get 0 by (27). □

9. A stylized model with very heavy tails; Sessions initiated at renewal times. See Mikosch and Resnick (2006).

9.1. The model.

$\{S_n, n \geq 1\}$ = times of session initiations; ordinary renewal process;

$$S_0 = 0,\ S_n = \sum_{i=1}^{n} X_i;\ \{X_n\} \text{ iid};$$
$$X_i \sim F_X(x);\ \bar{F}_X(x) = 1 - F_X(x) \in RV_{-\alpha}; 0 < \alpha < 1;$$

$\{L_n\}$ = session durations; iid non-negative rv's with common distribution $F_L(x)$ satisfying

$$\bar{F}_L(x) \in RV_{-\beta},\ 0 < \beta < 1;$$
$$\{L_n\} \perp\!\!\!\perp \{S_n\}.$$

$$M(t) = \text{number of active sessions at } t,$$
$$= \sum_{n=1}^{\infty} 1_{[S_n \leq t < S_n + L_n]}$$
$$A(t) = \int_0^t M(s)ds,$$
$$= \text{cumulative work inputted in } [0,t], \text{ assuming unit rate input.}$$

9.2. Cases.

1. Comparable tails: $\beta = \alpha$ and $\bar{F}_X(x) \sim c\,\bar{F}_L(x)$, $c > 0$, as $x \to \infty$.
   (a) The distribution tails of X_1 and L_1 are essentially the same.
   (b) For simplicity, we assume $c = 1$.
   (c) Kind of stability for M which converges weakly w/o normalization.
2. F_L heavier-tailed:
   (a) $0 < \beta < \alpha < 1$ or, if $\beta = \alpha$, then $\bar{F}_X(x)/\bar{F}_L(x) \to 0$ as $x \to \infty$ so that the distribution tail of X_1 is lighter than the distribution tail of L_1.
   (b) $0 = \beta < \alpha < 1$ so that the distribution tail of L_1 is slowly varying and thus again heavier than that of X_1.
   (c) Implies buildup in the M process.
3. F_X heavier-tailed: $\beta > \alpha$ so that the distribution tail of X_1 is heavier than the distribution tail of L_1.
   - Renewal epochs sparse relative to session lengths.
   - Of less interest.

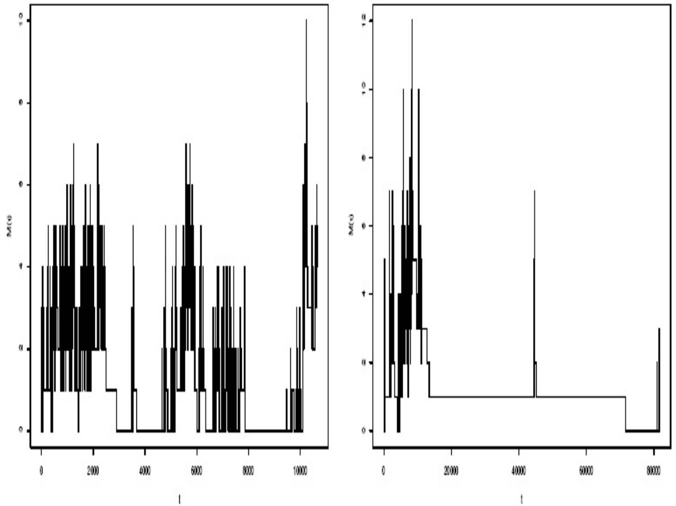

Figure 1: Paths of M; $\alpha = \beta = 0.9$ (left); $\alpha = \beta = 0.6$ (right).

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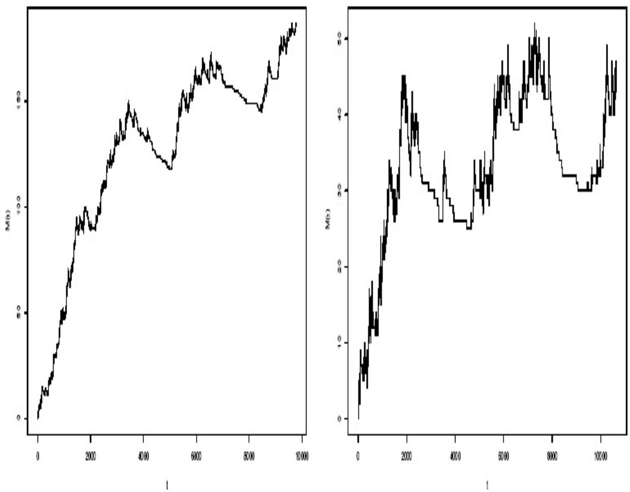

Figure 3. Session lengths tend to be longer than interarrival lengths; Paths of M; $(\alpha, \beta) = (0.9, 0.2)$ (left); $(\alpha, \beta) = (0.9, 0.4)$ (right).

9.3. Warm-up: Mean value analysis for $\alpha, \beta < 1$.

Obtain asymptotic behavior of $E\big(M(t)\big)$ from Karamata's Tauberian theorem. Let

$$U(x) = \sum_{n=0}^{\infty} F_X^{n*}(x), \quad x > 0, \quad = \text{renewal function.}$$

Since $0 < \alpha < 1$, well known (eg Feller, 1971); as $x \to \infty$,

$$U(x) \sim \big(\Gamma(1-\alpha)\,\Gamma(1+\alpha)\,\bar{F}_X(x)\big)^{-1} \sim c(\alpha)\,x^{\alpha}/\ell_F(x).$$

Therefore ($t \to \infty$),

$$\begin{aligned}
\mathbb{E}M(t) &= \sum_k P[S_k \le t < S_k + L_k] = \int_0^t U(dx)\,\bar{F}_L(t-x) \\
&= \int_0^1 \frac{\bar{F}_L(t(1-s))}{\bar{F}_L(t)} \frac{U(tds)}{U(t)} \Big(\bar{F}_L(t) U(t)\Big) \\
&\sim c(\alpha) \int_0^1 (1-s)^{-\beta} \alpha s^{\alpha-1} ds \, \frac{\bar{F}_L(t)}{\bar{F}_X(t)} = c'(\alpha) \frac{\bar{F}_L(t)}{\bar{F}_X(t)}.
\end{aligned}$$

$$\mathbb{E}M(t) \sim c'(\alpha)\frac{\bar{F}_L(t)}{\bar{F}_X(t)}, \quad (t \to \infty).$$

Conclusions

- Case (1): Comparable tails.

$$E\big(M(t)\big) \text{ converges to a constant.}$$

- Case (2): F_L is more heavy-tailed than F.

$$\mathbb{E}M(t) \to \infty.$$

- Case (3): F_X is more heavy-tailed than F_L.

$$\mathbb{E}M(t) \to 0.$$

and hence

$$M(t) \stackrel{L_1}{\to} 0.$$

so Case (3) may be of lesser interest. (Renewals are sparse relative to event durations that at any time there is not likely to be an event in progress.)

9.4. Summary of results.

Table 1: Limiting behavior of $M(t)$ as $t \to \infty$.

Conditions	Limit behavior of $M(t)$ as $t \to \infty$
$0 < \alpha < 1$ $\bar{F}_X \sim \bar{F}_L$	$M(t) \Rightarrow$ random limit.
$0 \le \beta < \alpha < 1$ or $0 < \alpha = \beta < 1$ and $\bar{F}_X = o(\bar{F}_L)$	$\frac{\bar{F}_X(t)}{\bar{F}_L(t)} M(t) \Rightarrow$ random limit.
$0 < \beta < 1$ $\mathbb{E}(X_1) < \infty$	$\frac{M(t)}{t\bar{F}_L(t)} \Rightarrow$ constant $\frac{M(t) - \text{random centering}}{\sqrt{t\bar{F}_L(t)}} \Rightarrow$ Gaussian rv
$0 < \beta \le \alpha = 1$ $\mathbb{E}(X_1) = \infty$	$\frac{M(t)}{t\bar{F}_L(t)\mu(t)} \Rightarrow$ constant $\mu(t) =$ truncated 1st moment X
$\mathbb{E}(X_1) < \infty$ $\mathbb{E}(L_1) < \infty$	Stationary version of $M(\cdot)$ exists

Focus on the first 2 rows corresponding to $\alpha,\ \beta < 1$.

9.5. Renewal: $\alpha = \beta < 1$, comparable tails.

A kind of stability exists for this case since fidi's of $M(t\cdot)$ converge in distribution to a limit.

9.5.1. Preliminaries

Define

$$N(x) = \sum_{n=0}^{\infty} 1_{[S_n \le x]} = \inf\{n : S_n > x\} = S^{\leftarrow}(x), \quad x \ge 0.$$

$$= \text{renewal counting function.}$$

$$\sum_k \epsilon_{(t_k, j_k)} = N_\infty = \text{ PRM(Leb} \times \nu_\alpha) \text{ on } [0,\infty) \times (0,\infty] := \mathbb{E}$$

$$\nu_\alpha(x,\infty] = x^{-\alpha}$$

$$X_\alpha(t) = \sum_{t_k \le t} j_k, \quad t \ge 0,$$

$$= \text{non-decreasing } \alpha\text{-stable Lévy motion with Lévy measure } \nu_\alpha.$$

$$b(t) \sim \left(\frac{1}{1-F_X}\right)^{\leftarrow}(t), \quad t \to \infty, \quad t\bar{F}_X(b(t)) \sim 1;$$

$$= \text{quantile function of } F_X$$

$$X^{(s)}(t) = \frac{S_{[st]}}{b(s)} \Rightarrow X_\alpha(t), \quad (s \to \infty),$$

in $D[0,\infty)$; ie, renewal epochs are asymptotically stable. Now invert:

$$(X^{(s)})^{\leftarrow} \Rightarrow X_\alpha^{\leftarrow} \quad (s \to \infty)$$

which gives after unpacking the inverse

$$\frac{1}{u} N(b(u)\cdot) \Rightarrow X_\alpha^{\leftarrow}(\cdot), \quad u \to \infty,$$
$$\frac{1}{u} \sum_{n=0}^{\infty} \epsilon_{\frac{S_n}{b(u)}} \Rightarrow X_\alpha^{\leftarrow} \quad \text{in } M_+[0,\infty).$$

Have not yet referenced L.

9.6. Comparable tails; $\alpha = \beta < 1$.

Define the time change map by

$$T : \mathbb{D}^{\uparrow}[0,\infty) \times M_+(\mathbb{E}) \mapsto M_+(\mathbb{E}), \qquad (\mathbb{E} = [0,\infty) \times (0,\infty])$$

by

$$T(x,m) = \tilde{m}$$

where $\tilde{m}$ is defined by

$$\tilde{m}(f) = \iint f(x(u),v)\, m(du,dv)\,, \quad f \in C_K^+(\mathbb{E})\,.$$

If m is a point measure with representation $m = \sum_k \epsilon_{(\tau_k, y_k)}$, then

$$T(x,m) = \sum_k \epsilon_{(x(\tau_k), y_k)}\,.$$

Steps for analysis: $\bar F_L \sim \bar F_X$; $\alpha=\beta$.

1. in $M_p(\mathbb{E})$, as $s\to\infty$, regular variation of $\bar F_L$ equiv to

$$\sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s},\frac{L_k}{b(s)})} \Rightarrow N_\infty = PRM(\text{Leb}\times\nu_\alpha).$$

2. Since $\{S_k\}$ is independent of $\{L_k\}$, get joint convergence in $D[0,\infty)\times M_p(\mathbb{E})$,

$$\Big(\frac{S_{[s\cdot]}}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s},\frac{L_k}{b(s)})}\Big) \Rightarrow (X_\alpha, N_\infty), \quad (s\to\infty)$$

3. Apply the a.s. continuous function T:

$$T\Big(\frac{S[s\cdot]}{b(s)}, \sum_{k=0}^{\infty} \epsilon_{(\frac{k}{s},\frac{L_k}{b(s)})}\Big) = \sum_{k=0}^{\infty} \epsilon_{(\frac{S_{[sk/s]}}{b(s)},\frac{L_k}{b(s)})} =$$

$$\sum_{k=0}^{\infty} \epsilon_{(\frac{S_k}{b(s)},\frac{L_k}{b(s)})} \Rightarrow T\big(X_\alpha, N_\infty\big).$$

4. Evaluate on $\{(u,v): u\le t\le u+v\}$ to get result for M: The fidi's of $M(t)$ satisfy as $s\to\infty$,

$$M(s\,t) = \sum_{k=0}^{\infty} 1_{[\frac{S_k}{s}\le t<\frac{S_k+L_k}{s}]} \Rightarrow M_\infty(t) = \sum_k 1_{[X_\alpha(t_k)\le t<X_\alpha(t_k)+j_k]}.$$

9.7. Case 2: $\bar{F}_L$ heavier; $\alpha, \beta < 1$.

Ingredients for analysis when $\bar{F}_X(t)/\bar{F}_L(t) \to 0$:

1. Recall $b(t)$ is the quantile function of F_X and satisfies
$$s\bar{F}_X(b(s)) \to 1, \quad (s \to \infty).$$
2. Since $\bar{F}_L \in RV_{-\beta}$,
$$\frac{s\,\bar{F}_X(b(s))}{\bar{F}_L(b(s))} F_L(b(s)\cdot) \xrightarrow{v} \nu_\beta.$$
in $M_+(0,\infty]$, where $\xrightarrow{v}$ denotes vague convergence.
3. Equivalent to previous convergence is (variant of basic convergence)
$$\frac{\bar{F}_X(b(s))}{\bar{F}_L(b(s))} \sum_{k=0}^{[s]} \varepsilon_{\frac{L_k}{b(s)}} \Rightarrow \nu_\beta\,.$$
4. Extend by adding time component:
$$\frac{\bar{F}_X(b(s))}{\bar{F}_L(b(s)} \sum_{k=0}^{\infty} \varepsilon_{(\frac{k}{s},\frac{L_k}{b(s)})} \Rightarrow \text{Leb} \times \nu_\beta\,.$$

5. Augment using independence of $\{L_n\}$ and $\{S_n\}$:

$$\left(\frac{S_{[s\cdot]}}{b(s)} \, , \frac{\bar{F}_X(b(s))}{\bar{F}_L(b(s))} \sum_{k=0}^{\infty} \varepsilon_{\left(\frac{k}{s}\, , \frac{L_k}{b(s)}\right)} \right) \Rightarrow (X_\alpha \, , \mathrm{Leb} \times \nu_\beta) \, .$$

6. Apply the a.s. continuous map T; evaluate the result on the correct region to get result for M: The fidi' s of M satisfy

$$\frac{\bar{F}_X(s)}{\bar{F}_L(s)} M(s\,t) \Rightarrow \int_0^t (t-u)^{-\beta} \, dX_\alpha^{\leftarrow}(u). \quad s \to \infty.$$

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