Parts 2-4: Regular Variation, Extreme Value Theory, Hidden Regular Variation, Conditioned Limit Laws

Sidney Resnick
School of Operations Research and Industrial Engineering
Rhodes Hall, Cornell University
Ithaca NY 14853 USA

http://legacy.orie.cornell.edu/∼sid
sir1@cornell.edu

May 19, 2008

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 1 of 71
Go Back
Full Screen
Close
Quit

1. Regularly varying functions and measures

A function $U : \mathbb{R}_+^d \mapsto \mathbb{R}_+$ is multivariate regularly varying if

$$\lim_{t\to\infty} \frac{U(t\mathbf{x})}{U(t\mathbf{1})} = \lambda(\mathbf{x}) \neq 0,$$

for $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$. If $d = 1$, limit must be a power function and we are dealing with functions U which are asymptotically like power functions; for $d = 1$,

$$\frac{U(tx)}{U(t)} \to x^\rho, \quad \rho \in \mathbb{R}.$$

Call ρ the index and when $d = 1$ we write $U \in RV_\rho$.
When $d > 1$, a scaling argument shows $\exists\ \rho \in \mathbb{R}$ and

$$\lambda(t\mathbf{x}) = t^\rho \lambda(\mathbf{x}),$$

and $U(t\mathbf{1}) \in RV_\rho$. Therefore, equivalent formulation is there exists $V \in RV_\rho$ such that

$$\frac{U(t\mathbf{x})}{V(t)} \to \lambda(\mathbf{x}) \neq 0,$$

or (sequential version when $\rho > 0$) $\exists\ b_n \to \infty$ such that

$$\frac{U(b_n\mathbf{x})}{n} \to \lambda(\mathbf{x}).$$

(Then if $V \uparrow$ can set $b_n = V^{\leftarrow}(n)$.)

Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

1.1. Connection to domain of attraction characterizations.

Suppose $\{X_n, n \geq 1\}$ are iid non-negative, common distribution function $F(x)$. The extreme is

$$M_n = \bigvee_{i=1}^{n} X_i = \max\{X_1, \dots, X_n\}.$$

One of the extreme value distributions is the Frechet:

$$\Phi_\alpha(x) := \exp\{-x^{-\alpha}\}, \quad x > 0, \ \alpha > 0.$$

Questions:

- What are conditions on F, called *domain of attraction conditions*, so that there exists $b_n > 0$ such that

  $$P[b_n^{-1} M_n \leq x] = F^n(b_n x) \to \Phi_\alpha(x). \tag{1}$$

  When (1) holds, we say F is in the domain of attraction of Φ_α and write $F \in MDA(\Phi_\alpha)$.

- How do you characterize the normalization sequence $\{b_n\}$?

Answers:

- DofA: One argues that we must have

$$x_0 = \sup\{x : F(x) < 1\} = \infty,$$

and Furthermore

$$b_n \to \infty.$$

In (1), take logarithms to get for

$$\lim_{n\to\infty} n(-\log F(b_n x)) = x^{-\alpha}, \quad x > 0.$$

Use

$$-\log(1-z) \sim z, \quad (z \to 0,$$

and (1) is equivalent to

$$\lim_{n\to\infty} n(1 - F(b_n x)) = x^{-\alpha}, \quad x > 0. \tag{2}$$

This is the sequential version of regular variation for $\bar{F} = 1 - F$.

- Characterize b_n: Set $U(x) = 1/(1 - F(x))$ and (2) is the same as

$$U(b_n x)/n \to x^\alpha, \quad x > 0,$$

and inverting, we find that

$$\frac{U^{\leftarrow}(ny)}{b_n} \to y^{1/\alpha}, \quad y > 0. \tag{3}$$

So $U^{\leftarrow}(n) = (1/(1-F))^{\leftarrow}(n) \sim b_n$ and this determines b_n by the convergence to types theorem.

1.2. Conclusion: Connecting regular variation and domains of attraction in one dimension.

With

$$\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}, \quad x > 0,\ \alpha > 0,$$

we have

$$F \in MDA(\Phi_\alpha) \text{ iff } \lim_{t\to\infty} \frac{\bar F(tx)}{\bar F(t)} = x^{-\alpha},\ x > 0;$$

that is, $\bar F \in RV_{-\alpha}$.

1.3. Behavior of one dimensional regularly varying functions:

Regularly varying functions behave asymptotically like power functions. Helpful notation: Call $L(x)$ slowly varying if $L(\cdot) \in RV_0$. Then if

$$U \in RV_\rho$$

we have

$$L(x) := U(x)/x^\rho \in RV_0$$

and we can write

$$U(x) = x^\rho L(x).$$

Rules for manipulating:

- Karamata theorem: For $\rho > -1$,

$$\int_0^x U(t)dt$$

behaves as if $L(t)$ comes out of the integral and the power part integrates. So if $U(x) = x^\rho L(x)$, then

$$\begin{aligned}\int_0^x U(t)dt =& \int_0^x t^\rho L(t)dt \\ \sim& L(x) \int_0^x t^\rho dt = L(x)\frac{x^{\rho+1}}{\rho+1} \\ =& \frac{xU(x)}{\rho+1}.\end{aligned}$$

- Differentiation: If $U \in RV_\rho$ has a monotone density $u(x)$, then $u(x) \in RV_{\rho-1}$ (as if it were a power function).

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Go Back
Full Screen
Close
Quit

- Regularly varying functions have smooth asymptotically equivalent versions which comes from the *Karamata representation*: if $U \in RV_\rho$,

  $$U(x) = c(x)\exp\{\int_1^x \frac{\rho(s)}{s} ds\},$$

  where

  $$c(x) \to c_0, \quad \rho(t) \to \rho.$$

- The regular variation ratio converges locally uniformly.

1.4. Multivariate Regular Variation for Multivariate Distribution Functions and Measures

Application to distributions: Let $\boldsymbol{Z}$ be a random vector in $\mathbb{R}_+^d$ with df F. A *regularly varying tail* means

$$\frac{1 - F(t\mathbf{x})}{1 - F(t\mathbf{1})} \to \nu_*([\mathbf{0}, \mathbf{x}]^c),$$

for some Radon measure ν_*. Awkward to deal with multivariate df's and better to deal with measures.

Let

$$\begin{aligned}\mathbb{E} =&[0,\infty]^d \setminus \{\mathbf{0}\}\\ \aleph =&\{\mathbf{x}\in\mathbb{E} : \|\mathbf{x}\| = 1\},\\ R =&\|\boldsymbol{Z}\|, \quad \boldsymbol{\Theta} = \frac{\boldsymbol{Z}}{\|\boldsymbol{Z}\|}\in\aleph.\end{aligned}$$

The following are equivalent.

1. $\exists$ a Radon measure ν_* on $\mathbb{E}$ such that
$$\begin{aligned}\lim_{t\to\infty}\frac{1-F(t\mathbf{x})}{1-F(t\mathbf{1})} =&\lim_{t\to\infty}\frac{\mathbb{P}\big[\frac{\boldsymbol{Z}_1}{t}\in[\mathbf{0},\mathbf{x}]^c\big]}{\mathbb{P}\big[\frac{\boldsymbol{Z}_1}{t}\in[\mathbf{0},\mathbf{1}]^c\big]}\\ =&c\nu_*\Big([\mathbf{0},\mathbf{x}]^c\Big),\end{aligned}$$
some $c>0$ and for all points $\mathbf{x}\in[\mathbf{0},\boldsymbol{\infty})\setminus\{\mathbf{0}\}$ which are continuity points of $\nu_*([\mathbf{0},\cdot]^c)$.
2. $\exists$ a function $b(t)\to\infty$ and a Radon measure ν_* on $\mathbb{E}$ such that in $M_+(\mathbb{E})$
$$t\mathbb{P}\Big[\frac{\boldsymbol{Z}}{b(t)}\in\cdot\Big]\xrightarrow{v}\nu_*,\quad t\to\infty.$$
3. $\exists$ a pm $S(\cdot)$ on $\aleph$ and $b(t)\to\infty$ such that
$$t\mathbb{P}\Big[\Big(\frac{R}{b(t)},\boldsymbol{\Theta}\Big)\in\cdot\Big]\xrightarrow{v}c\nu_\alpha\times S$$

in $M_+(\big((0,\infty] \times \aleph\big)$, where $c > 0$.

Notes:

- Can replace function $b(t)$ by sequence $b(n)$.
- $\xrightarrow{v}$ means vague convergence defined as follows: Let $M_+(\mathbb{E})$ be the Radon measures on $\mathbb{E}$. (Radon means the measure is finite on relatively compact sets.) $M_+(\mathbb{E})$ can be metrized by vague convergence: Let $\mu_n(\cdot), n \geq 0$ be measures in $M_+(\mathbb{E})$. Then

  $$\mu_n \xrightarrow{v} \mu_0$$

  iff

  $$\mu_n(f) := \int_E f d\mu_n \to \int_E f d\mu =: \mu(f) \quad (n \to \infty)$$

  for all non-negative, continuous functions with compact support on $\mathbb{E}$.
- Generally, vague convergence can be reduced to convergence of measures on a class of rectangles suited to the compact sets of $\mathbb{E}$.

1.5. Statistical difficulty.

- This formulation is good for theory but bad for applications.
- Being able to norm each component by the same $b(t)$ means marginal tails are the same–almost never happens in practice. Multivariate data from a distribution with heavy tailed marginals, never have the same α's.

  Set

  $$\boldsymbol{Z} = (Z^{(1)}, \dots, Z^{(d)}).$$

  Norming each component with the same $b(t)$ means

  $$\mathbb{P}[Z^{(i)} > x] \sim c_{ij}\mathbb{P}[Z^{(j)} > x], \quad x \to \infty.$$

  and if $c_{ij} > 0$, then the tail index of $Z^{(i)}$ and $Z^{(j)}$ are the same.

  Examples:

  - Xchr vs USD of (FR, JAP).
  - (Size of document downloaded, download time).
  - (log return IBM, log return P&G).

1.5.1. More flexibility.

A more flexible defn of a multivariate heavy tail is: $\exists\ b_n^{(i)} \to \infty$ for $i = 1, \dots, d$ and $\exists$ Radon ν such that

$$n\mathbb{P}\Big[\Big(\frac{Z^{(i)}}{b_n^{(i)}}, i = 1, \dots, d\Big) \in \cdot\ \Big] \to \nu. \tag{4}$$

Theorem. If (4) and

$$n\mathbb{P}\Big[\frac{Z^{(i)}}{b_n^{(i)}} \in \cdot\ \Big] \to \nu_{\alpha_i}, \quad \nu_{\alpha_i}(x, \infty] = x^{-\alpha_i},\ x > 0, \alpha_i > 0,\ \forall i,$$

then

$$nF_*(n\cdot\,) = n\mathbb{P}\Bigg[\Bigg(\frac{\frac{1}{1-F_{(i)}(Z^{(i)})}}{n}, i = 1, \dots, d\Bigg) \in \cdot\ \Bigg] \to \nu_*(\cdot)$$

where ν_* is *standard*; that is, Radon and

$$\nu_*(tA) = t^{-1}\nu_*(A).$$

Note if for $i = 1, \dots, d$

$$1 - F_{(i)}(x) \sim x^{-\alpha_i}, \quad x \to \infty,$$

then

$$n\mathbb{P}\Big[\Big(\frac{(Z^{(i)})^{\alpha_i}}{n},\ i = 1, \dots, d\Big) \in \cdot\ \Big] \to \nu_*(\cdot).$$

1.5.2. How to get the standard case in practice:

(a) Simple minded. Hope (assume, pray) $1 - F_{(i)}(x) \sim x^{-\alpha_i}$ for all i and then power up. (Slow variation hard (impossible?) to detect in practice and this works reasonably.)

BUT: Must estimate α's. (Ouch!)

(b) Use ranks method.

BUT: Lose independence among observations.

1.6. Significance of the limit measure.

The limit measure ν_* controls the (asymptotic) dependence structure.

1.6.1. Asymptotic independence.

The distribution F of $\boldsymbol{Z}$ possesses the property of *asymptotic independence* if

1. $\nu_*\big((\mathbf{0},\boldsymbol{\infty})\big)=0$ so that ν_* concentrates on the axes;
   OR
2. S concentrates on $\{\boldsymbol{e}_i, i=1,\dots,d\}$.

For $d=2$ this means

$$\begin{aligned}\mathbb{P}[Z^{(2)}>t|Z^{(1)}>t]&=\frac{\mathbb{P}[Z^{(2)}>t,Z^{(1)}>t]}{\mathbb{P}[Z^{(1)}>t]}\\ &\to(const)\nu_*\Big(\mathbf{1},\boldsymbol{\infty}\Big]=0.\end{aligned}$$

Remarks:

- The probability of both components being large is negligible.
- The concept was invented to answer the following query: Let $\{\boldsymbol{Z}_j, j \geq 1\}$ be iid, non-negative random vectors in $\mathbb{R}^d_+$ with common distribution F. The necessary and sufficient condition for there to exist $b(n) \to \infty$ such that

  $$P[\bigvee_{i=1}^{n} \frac{Z_i^{(l)}}{b(n)} \leq x^{(l)}, l = 1, \dots, d] \to G(\mathbf{x})$$

  is $1 - F$ is multivariate regularly varying with the original definition. The necessary and sufficient condition for G to be a product distribution is asymptotic independence.

1.6.2. Asymptotic dependence.

The distribution F of $\boldsymbol{Z}_1$ possesses the property of *asymptotic dependence* if

1. ν_* concentrates on $\{t\frac{\mathbf{1}}{\|\mathbf{1}\|} : t > 0\}$, the diagonal line,

   or

2. S concentrates on $\{\frac{\mathbf{1}}{\|\mathbf{1}\|}\}$.

Means:

- $$\mathbb{P}[Z^{(2)} > t | Z^{(1)} > t] \to 1.$$

- $$G(\mathbf{x}) = G(\wedge_{i=1}^{d} x^{(i)}),$$

  the distribution of a random vector all of whose componenets are equal.

1.7. Formulate the domain of attraction problem in Multivariate EVT

By analogy with the one dimensional problem discussed in relation to regular variation, we consider the following problem(s): Given $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ iid random vectors in $\mathbb{R}^d$ with common distribution F.

1.7.1. Problems:

- When do there exist

  $$\boldsymbol{a}(n) = (a^{(1)}(n), \dots, a^{(d)}(n)) \in \mathbb{R}^d_+, \quad \boldsymbol{b}(n) = (b^{(1)}(n), \dots, b^{(d)}(n)) \in \mathbb{R}^d,$$

  and a probability distribution G such that

  $$\begin{aligned} P[(\bigvee_{j=1}^{n} \boldsymbol{X}_j - \mathbf{b}(n))/\boldsymbol{a}(n) \le \mathbf{x}] &= F^n(\boldsymbol{a}(n)\mathbf{x} + \mathbf{b}(n)) \\ &= P[(\bigvee_{j=1}^{n} X_j^{(i)} - b^{(i)}(n))/a^{(i)}(n) \le x^{(i)}; i = 1, \dots, d] \to G(\mathbf{x})? \end{aligned} \tag{5}$$

- What is the family of possible limits G?
- For a given G how do you characterize $\boldsymbol{a}(n)$ and $\boldsymbol{b}(n)$?

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 16 of 71
Go Back
Full Screen
Close
Quit

- For a given G in the family of possible limits, what properties does F have to satisfy in order for (5) to hold.

  If (5) holds, we say F is in the domain of attraction of G and write $F \in MDA(G)$.

1.7.2. Some superficial and cheap answers:

- Joint convergence implies marginal convergence:

$$F_i^n\big(a^{(i)}(n)x^{(i)} + b^{(i)(n)}\big) \to G_i(x^{(i)}), \quad (n \to \infty).$$

  So if we know how to find normalizing constants in one dimension, we can find them in d-dimensions.

- From (5), we take logarithms to get

$$n\Big(1 - F\big(\boldsymbol{a}(n)\mathbf{x} + \boldsymbol{b}(n)\big)\Big) \to -\log G(\mathbf{x}), \quad (G(\mathbf{x}) > 0).$$

  Re-write this as

$$nP\Big\{\Big[\frac{\boldsymbol{X}_1 - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \le \mathbf{x}\Big]^c\Big\} \to -\log G(\mathbf{x}),$$

  or

$$nP\Big[\frac{\boldsymbol{X}_1 - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \cdot\Big] \xrightarrow{v} \nu(\cdot),$$

where

$$\nu([-\boldsymbol{\infty},\mathbf{x}]^c) = -\log G(\mathbf{x}).$$

The measure ν is called the *exponent measure* of G or the *limit measure*, since

$$G(\mathbf{x}) = \exp\{-\nu([-\boldsymbol{\infty},\mathbf{x}]^c)\}.$$

2. Regular variation on cones

Suppose $\mathbb{CONE}$ is a cone centered at $\mathbf{0}$:

$$\mathbf{x} \in \mathbb{CONE} \quad \Rightarrow t\mathbf{x} \in \mathbb{CONE}, \qquad t > 0.$$

Suppose $\boldsymbol{Z}^*$ is a random vector. $\boldsymbol{Z}^*$ has a regularly varying distribution in standard form on $\mathbb{CONE}$ if

$$tP\left[\frac{\boldsymbol{Z}^*}{t} \in \cdot\,\right] \xrightarrow{v} \nu^*(\cdot), \qquad \text{in } M_+(\mathbb{CONE}).$$

Cone $\mathbb{CONE}$	Application
$\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$	multivariate extreme value theory
$\mathbb{E}_0 = (\mathbf{0}, \boldsymbol{\infty}]$	hidden regular variation, coefficient of tail dependence;
$\mathbb{E}_\sqcap = [0, \infty] \times (0, \infty]$	Conditioned limit theorems when one component is extreme.
$[-\boldsymbol{\infty}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$	weak conv to stable laws

Table 1: Theories stemming from standard multivariate regular variation on different cones.

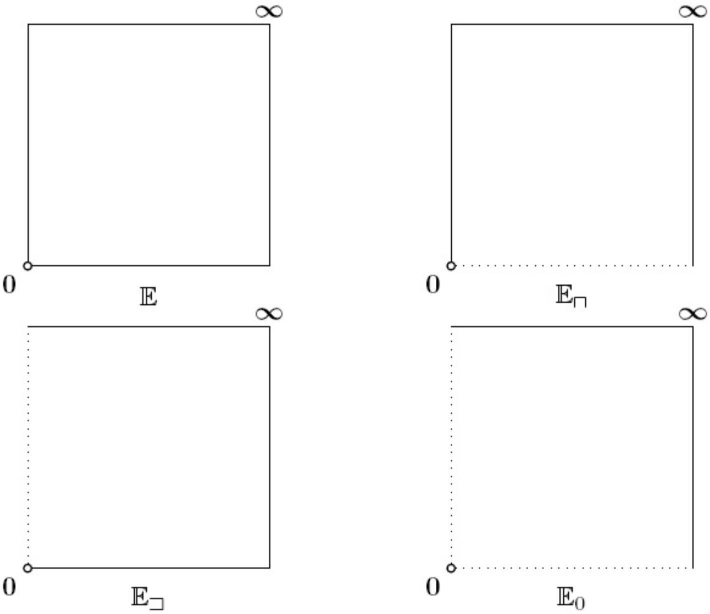


FIGURE 1. The different cones in 2-dimensions

Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 20 of 71
Go Back
Full Screen
Close
Quit

2.1. The cone $\mathbb{E} = [0,\infty] \setminus \{0\}$ and EVT.

Let $\boldsymbol{Z}^* = (Z^{*(1)}, \dots, Z^{*(d)})$ be standard regularly varying random vector on $\mathbb{E}$:

$$tP\Big[\frac{\boldsymbol{Z}^*}{t} \in \cdot\,\Big] \xrightarrow{v} \nu^*(\cdot).$$

Suppose $(b^{(i)}, a^{(i)})$, $i = 1 \dots d$ satisfy

$$\frac{b^{(i)}(tx) - b^{(i)}(t)}{a^{(i)}(t)} \to \psi_i(x) \not\equiv 0, \quad x > 0,\ t \to \infty, \tag{6}$$

and that each $b^{(i)}$ is non-decreasing. In inverted form (6) is

$$b^{(i)\leftarrow}\big(a^{(i)}(t)z + b^{(i)}(t)\big)/t \to \psi_i^{\leftarrow}(z), \quad t \to \infty. \tag{7}$$

Then based on standard regular variation

$$\begin{aligned}
&tP\Big\{\Big[\frac{b^{(i)}\big(Z^{*(i)}\big) - b^{(i)}(t)}{a^{(i)}(t)} \le x^{(i)}\,,\ i = 1 \dots d\Big]^c\Big\}\\
=&tP\Big\{\Big[\frac{Z^{*(i)}}{t} \le b^{(i)\leftarrow}\big(a^{(i)}(t)x^{(i)} + b^{(i)}(t)\big)/t\,,\ i = 1 \dots d\Big]^c\Big\}\\
\to&\nu_*\big(\{\boldsymbol{y} : y^{(i)} \le \psi_i^{\leftarrow}(x^{(i)})\,,\ i = 1 \dots d\}^c\big)\,.
\end{aligned} \tag{8}$$

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 21 of 71
Go Back
Full Screen
Close
Quit

Conclusions:

- Given $\boldsymbol{Z}^*$ which is standard regularly varying on $\mathbb{E}$ with limit measure ν^*, we have
$$\boldsymbol{X} = \big(b^{(i)}(Z^{*(i)}),\, i = 1, \dots, d\big)$$
in the multivariate domain of attraction of an EV distribution G with exponent measure
$$\nu^*\big(\{\boldsymbol{y} : y^{(i)} \le \psi_i^{\leftarrow}(x^{(i)})\,,\; i = 1 \dots d\}^c\big)\,.$$
- Vice versa: Given a random vector $\boldsymbol{X}$ in the domain of attraction of the multivariate EV distribution $G(\mathbf{x})$, there exist monotone transformations $b^{(i)}(t)$, $i = 1, \dots, d$ such that
$$\boldsymbol{Z}^* = \big((b^{(i)})^{\leftarrow}(X^{(i)}),\, i = 1, \dots, d\big)$$
is standard regularly varying.

Where do $b^{(i)}$ and $a^{(i)}$ come from?

The equations (6) and (7) are the one-dimensional domain of attraction conditions and say

$$F_i(x) := P[X^{(i)} \leq x] \in MDA(G_{\gamma_i}), \quad \gamma_i \in \mathbb{R} \quad i = 1, \dots, d$$

with

$$b^{(i)}(t) = \left(\frac{1}{1 - F_i}\right)^{\leftarrow}(t), \quad i = 1, \dots, d$$

and are equivalent to

$$t\big(1 - F_i(a^{(i)}(t)x + b^{(i)}(t))\big) \to -\log G_{\gamma_i}(x) \quad i = 1, \dots, d.$$

and comparing with (7) we see

$$\psi_i^{\leftarrow}(z) = \frac{1}{-\log G_{\gamma_i}(z)}.$$

3. Standardization

Standardization is the process of marginally transforming

$$\boldsymbol{X} \mapsto \boldsymbol{Z}^*$$

so that the distribution of $\boldsymbol{Z}^*$ is standard regularly varying on a cone $\mathbb{CONE}$: For some Radon measure $\nu^*(\cdot)$

$$tP\left[\frac{\boldsymbol{Z}^*}{t} \in \cdot\right] \xrightarrow{v} \nu^*(\cdot), \qquad \text{in } M_+(\mathbb{CONE}).$$

For EVT,

$$\mathbb{CONE} = \mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}.$$

In general, depending on the cone, this says one or more components of $\boldsymbol{Z}^*$ are asymptotically Pareto. For the EVT case, each is asymptotically Pareto.

3.1. Theoretical advantages of standardization:

- Standardization is analogous to the copula transformation but is better suited to studying limit relations (Klüppelberg and Resnick, 2008).
- In Cartesian coordinates, the limit measure has scaling property:

$$\nu^*(c\,\cdot\,) = c^{-1}\nu^*(\cdot), \quad c > 0.$$

- The scaling in Cartesian coordinates allows transformation to polar coordinates to yield a product measure: An angular measure exists allowing characterization of limits:

$$\nu^*\{\mathbf{x} : \|\mathbf{x}\| > r, \frac{\mathbf{x}}{\|\mathbf{x}\|} \in \Lambda\} = r^{-1}S(\Lambda),$$

  for Borel subsets Λ of the unit sphere in $\mathbb{CONE}$.

- For EVT, S is a finite measure (wlog taken to be a pm) but when $\mathbb{CONE} = \mathbb{E}_0$ or $\mathbb{CONE} = \mathbb{E}_{\sqcap}$, S is NOT necessarily finite.
- See de Haan and Resnick (1977), Resnick (1987), Mikosch (2005, 2006), de Haan and Ferreira (2006), Resnick (2007).

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

4. Reduction of multivariate regular variation to a one-dimensional criterion.

For a general cone $\mathbb{CONE}$, one can hope to dream up variants of the Cramer-Wold device to reduce multivariate regular variation to one dimensional regular variation.

Example: EVT

For EVT, $\mathbb{CONE} = \mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$. Then

$$tP[\frac{\boldsymbol{Z}^*}{t} \in \cdot] \xrightarrow{v} \nu^*(\cdot), \qquad \text{in } M_+(\mathbb{E})$$

iff

$$\forall \boldsymbol{s} \geq 0,\ \boldsymbol{s} \neq 0,$$

$$\lim_{t\to\infty} tP[\bigvee_{i=1}^{d} s^{(i)} Z^{*(i)} > tx] = c(\boldsymbol{s})x^{-1}, \quad x > 0.$$

See de Haan (1978).

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 26 of 71
Go Back
Full Screen
Close
Quit

5. Asymptotic Independence in EVT

If (X, Y) is in a bivariate domain of attraction of a multivariate extreme value distribution $G(x, y)$,

$$tP\Big\{\Big[\frac{X-\beta(t)}{\alpha(t)} \le x, \frac{Y-b(t)}{a(t)} \le y\Big]^c\Big\} \to -\log G(x,y), \quad t \to \infty,$$

then *asymptotic independence* of (X, Y) means

$$tP\Big[\frac{X-\beta(t)}{\alpha(t)} > x, \frac{Y-b(t)}{a(t)} > y\Big] \to 0, \quad t \to \infty.$$

This roughly says, (de Haan and Ferreira, 2006, Resnick, 1987)

- The probability of both X and Y being biggish is smallish.
- Componentwise maxima (normalized) of iid samples of (X, Y) are asymptotically distributed as a product measure.

If $\boldsymbol{Z}^* \in \mathbb{R}^d_+$ has a standard regularly varying distribution, then asymptotic independence means

$$tP\Big[\frac{\boldsymbol{Z}^*}{t} \in \cdot\,\Big] \xrightarrow{v} \nu^*(\cdot)$$

where

$$\nu^*\{\mathbf{x} \in \mathbb{E} : x^{(i)} \wedge x^{(j)} > 0, \text{ some } i, j\} = 0.$$

6. Examples

6.1. Asymptotic independence but no hidden regular variation

Let $U \sim U(0,1)$ and

$$(X,Y) = \Big(\frac{1}{U}, \frac{1}{1-U}\Big).$$

Then (X,Y) has a standard regularly varying distribution which possesses asymptotic independence.
Reduction to 1-dimensional criterion:

$$P[X \bigvee Y > x] = \frac{2}{x}, \quad x > 2.$$

Note: we have not checked ALL max-linear combinations BUT

- this is representative calculation,
- This suggests what is likely to be checked in practice.

6.2. Example: Same rv's; different variation on different cones.

Let X and Z be iid Pareto(1) random variables and define

$$Y = X^2 \wedge Z^2.$$

Then in $\mathbb{E}, \mathbb{E}_0$ and $\mathbb{E}_\sqcap$ check convergence on representative relatively compact sets:

- In $M_+(\mathbb{E})$, asymptotic independence,

$$t\mathbb{P}\left[\left(\frac{X}{t}, \frac{Y}{t}\right) \in ([0,x] \times [0,y])^c\right] \to \frac{1}{x} + \frac{1}{y}, \quad x \vee y > 0.$$

- In $M_+(\mathbb{E}_0)$:

$$t\mathbb{P}\left[\left(\frac{X}{t^{2/3}}, \frac{Y}{t^{2/3}}\right) \in (x,\infty] \times (y,\infty]\right] \to \frac{1}{x\sqrt{y}}, \quad x \wedge y > 0,$$

  or in standard form,

$$t\mathbb{P}\left[\left(\frac{X^{3/2}}{t}, \frac{Y^{3/2}}{t}\right) \in (x,\infty] \times (y,\infty]\right] \to \frac{1}{x^{2/3}y^{1/3}}, \quad x \wedge y > 0.$$

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

- In $M_+(\mathbb{E}_\sqcap)$:

$$t\mathbb{P}\Big[\Big(\frac{X}{t^{1/2}},\frac{Y}{t}\Big)\in[0,x]\times(y,\infty]\Big]\to\frac{1}{y}-\frac{1}{\sqrt{y}}\times\frac{1}{x\vee\sqrt{y}},\qquad x\geq 0, y>0,$$

or in standard form,

$$t\mathbb{P}\Big[\Big(\frac{X^2}{t},\frac{Y}{t}\Big)\in[0,x]\times(y,\infty]\Big]\to\frac{1}{y}-\frac{1}{\sqrt{y}}\times\frac{1}{\sqrt{x}\vee\sqrt{y}},\qquad x\geq 0, y>0.$$

Angular measures

For these examples we get the angular measure after polar co-ordinate transformation

$$(x,y)\mapsto(r,\theta):=(x+y,\frac{x}{x+y}).$$

- On $\mathbb{E}$,

$$S(d\theta)=\frac{1}{2}[\delta_{\{0\}}(d\theta)+\delta_{\{1\}}(d\theta)].$$

- For the standard form on $\mathbb{E}_0$ we have the angular measure

$$S(d\theta)=\frac{1}{\theta^{5/3}(1-\theta)^{4/3}}d\theta,\qquad 0<\theta<1.$$

The density is not integrable near 0 or 1 so S is not a finite measure on $(0,1)$.

- For the standard form on $\mathbb{E}_{\sqcap}$

$$S(d\theta)=\begin{cases}\theta^{-\frac{3}{2}}(1-\theta)^{-\frac{3}{2}}d\theta, & \frac{1}{2}\le\theta<1\\ 0 & 0\le\theta<\frac{1}{2}.\end{cases}$$

This is not integrable near 1 so S is not finite on $[0,1)$.

6.3. Angular measure finite or infinite?

Example of class of examples of standard regular variation on $\mathbb{E}_{\sqcap}$ indexed by distributions on $[0,\infty]$. Suppose

$$R\sim \text{Pareto}(1)\text{ on }[1,\infty),\qquad \theta\ge 0,\theta\sim G(\cdot)\text{ on }[0,\infty].$$

Assume $R \perp\!\!\!\perp \theta$. Define

$$(X,Y)=(R\theta,R).$$

Therefore we have for $y>0,x\ge 0$ (and $ty>1$),

$$t\mathbb{P}\Big(\frac{X}{t}\le x,\frac{Y}{t}>y\Big)=t\mathbb{P}\Big(\frac{R\theta}{t}\le x,\frac{R}{t}>y\Big)=t\int_{ty}^{\infty}\mathbb{P}(\theta\le\frac{tx}{r})r^{-2}dr$$

$$=\frac{1}{x}\int_{0}^{x/y}G(s)ds=\mu\big([0,x]\times(y,\infty]\big).$$

Polar co-ordinates: The the angular measure $S(\cdot)$ on $\mathbb{E}_{\sqcap}$ for $0 \leq \eta < 1$ is given by

$$S([0,\eta]) = \mu\{(u,v) : u+v > 1, \frac{y}{u+v} \leq \theta\}.$$

Hence

$$\begin{aligned} tP\left[\frac{X+Y}{t} > 1, \frac{X}{X+Y} \leq \eta\right] &= tP\left[\frac{R\theta+R}{t} > 1, \frac{R\theta}{R\theta+R} \leq \eta\right] \\ &= tP\left[\frac{R(1+\theta)}{t} > 1, \theta \leq \frac{\eta}{1-\eta}\right] \\ &= t\int_{0\leq s\leq\frac{\eta}{1-\eta}} P\left[\frac{R}{t}(1+s) > 1\right] G(ds) \\ &\to \int_{0\leq s\leq\frac{\eta}{1-\eta}} (1+s)G(ds) =: S[0,\eta]. \end{aligned}$$

Hence

$$S([0,\eta]) = \int\limits_{0\leq s\leq\frac{\eta}{1-\eta}} (1+s)G(ds), \quad 0 \leq \eta < 1.$$

Conclude: S is a finite angular measure on $[0,1)$ if and only if G has first moment.

7. How to Standardize?

Theoretical formulations often assume the *standard case*.

- Standard case almost never happens in practice.
- A vector which is standard regularly varying has each component having the same (asymptotically equivalent) tail.

How to transform to the standard case in practice?

- In heavy tail analysis, the simplest method: Hope $1 - F_{(i)}(x) \sim x^{-\alpha_i}$ for all i and then power up.
  BUT: Must estimate α's.
- More generally, for EVT: estimate marginals (somehow) and transform using the marginals.
  BUT: Difficult to quantify the error made when estimating marginal

  distributions.
- Use ranks method (Huang, 1992; de Haan & de Ronde).
  BUT: Lose independence among observations.

The ranks method:

Given d-dimensional random vectors $\{\boldsymbol{X}_1, \dots, \boldsymbol{X}_n\}$ where

$$\boldsymbol{X}_i = (X_i^{(1)}, \dots, X_i^{(d)}), \quad i = 1, \dots, n,$$

define the (anti)-ranks for each component: Comparing the jth components, $X_1^{(j)}, \dots, X_n^{(j)}$, the anti-rank of $X_i^{(j)}$ is

$$\begin{aligned} r_i^{(j)} &= \sum_{l=1}^{n} 1_{[X_l^{(j)} \geq X_i^{(j)}]} \\ &= \#\text{ jth components} \geq X_i^{(j)}. \end{aligned}$$

Replace each $\boldsymbol{X}_i$ by

$$\boldsymbol{X}_i \mapsto \big(1/r_i^{(j)},\ j = 1, \dots, d\big) =: \boldsymbol{Z}^*.$$

Allows

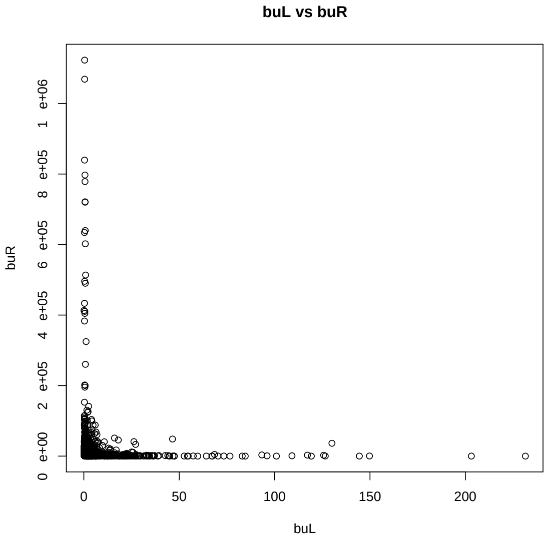
- Estimation of the angular measure S.
- Estimation of the hidden angular measure in hidden regular variation (Heffernan and Resnick, 2005).

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 34 of 71
Go Back
Full Screen
Close
Quit

8. Data examples

8.1. BuL vs BuR Scatterplot:

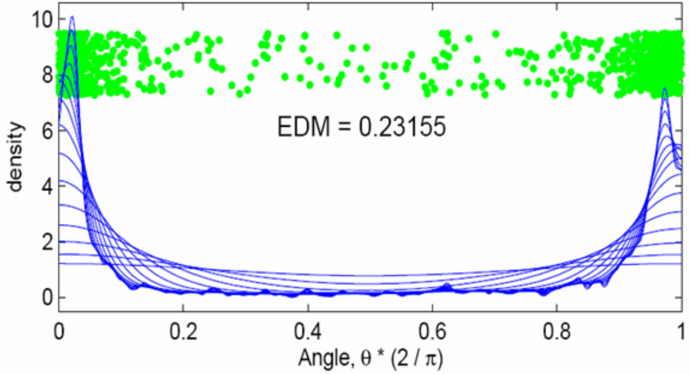
Data processed from the original 1995 Boston University data; 4161 file sizes (F) and download times (L) noted and transmission rates (R) inferred. The data consists of bivariate pairs (R,L).



8.2. UNC

Steps:

- Transform (F,R) data using rank method.
- Convert to polar coordinates.
- Keep 2000 pairs with biggest radius vector.
- Compute density estimate for angular measure S.


EDM = 0.23155
density
0
2
4
6
8
10
0
0.2
0.4
0.6
0.8
1
Angle, θ * (2 / π)

8.3. The Auckland-II trace

(`http://pma.nlanr.net/traces/long/auck2.html`) a collection of long GPS-synchronized IP header traces captured at the University of Auckland Internet uplink since 1999.

- Connection level data characterized by packet headers.
- connection = 5 tuple (source ip, dest ip, source port, dest port, protocol)
- Amalgamate the data into clusters in hopes that cluster heads are approx Poisson.
  - No unique way to do this.
  - One proposal:
    - Organize packets into e2e streams–packets with same ip source and destination. Con: ignores application.
    - Create sessions by clustering e2e streams according to the rule that 2 consecutive packets part of the same cluster if time separation between them is below a threshold.
    - For a session, compute total payload (F), duration (L) and then (average) rate (R)
    - Compute a peak rate for each session by chopping session durations into small bins, computing the average rate per bin, and taking max of these rates.

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

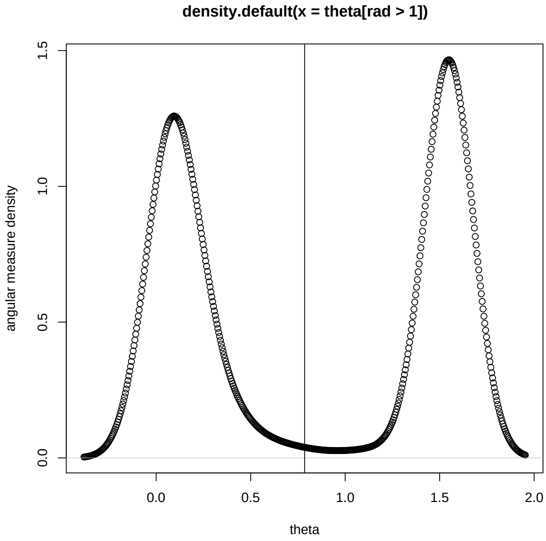


Figure 1: k=1500, Angular measure F,R; length=54343

9. Hidden Regular Variation.

9.1. Why asymptotic independence creates problems.

- Estimators of various parameters may behave badly under asymptotic independence; eg, estimator of the spectral measure S. Estimators may be asymptotically normal with an asymptotic variance of 0 (oops!).

- Estimators of probabilities given by asymptotic theory may be uninformative.

Scenario: Estimate the probability of simultaneous non-compliance.

Supppose $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$ = concentrations of different pollutants. Environmental agencies set critical levels $\boldsymbol{t}_0 = (t_0^{(1)}, t_0^{(2)})$ which not be exceeded. Imagine simultaneous *non-compliance* creates a health hazard. Worry about

$$[\text{ health hazard }] = [\boldsymbol{Z} > \boldsymbol{t}_0] = [Z^{(j)} > t_0^{(j)};\ j = 1, 2].$$

Asymptotic independence might lead one to report

$$P[\text{ health hazard }] = P[\boldsymbol{Z} > \boldsymbol{t}_0] = 0.$$

9.2. A submodel of asymptotic independence.

The random vector $\boldsymbol{Z}$ has a distribution possessing *hidden regular variation* if

1. Regular variation on the big cone $\mathbb{E} = [0,\infty]^2 \setminus \{\mathbf{0}\}$:

   $$t\mathbb{P}[\frac{\boldsymbol{Z}}{b(t)} \in \cdot] \xrightarrow{v} \nu,$$

   (if $\boldsymbol{Z}$ is standard, $b(t) \sim t$)

AND

2. Regular variation on the small cone $\mathbb{E}_0 = (0,\infty]^2$: $\exists$ a non-decreasing function $b^0(t) \uparrow \infty$ such that

   $$b(t)/b^0(t) \to \infty$$

   and $\exists$ a measure $\nu^0 \neq 0$ which is Radon on $\mathbb{E}_0 = (0,\infty]^2$ and such that

   $$tP[\frac{\boldsymbol{Z}}{b^0(t)} \in \cdot] \xrightarrow{v} \nu^0 \;=\; \text{hidden measure}$$

   on the cone $\mathbb{E}_0$.

Then there exists $\alpha^0 \geq \alpha$ such that $b^0 \in RV_{1/\alpha^0}$.

Consequences:

- With the right formulation,

$$\begin{aligned}&\text{Second order regular variation} + \text{asy indep}\\ &\quad\Rightarrow \text{hidden regular variation}\\ &\quad\Rightarrow \text{asymptotic independence.}\end{aligned}$$

- Reduction to one-dimensional regular variation: Hidden regular variation means for every $\boldsymbol{s} \geq 0$, $\boldsymbol{s} \neq 0$, $\bigvee_{i=1}^{d} s^{(i)} Z^{(i)}$ has distribution with a regularly varying tail of index α and for every $\boldsymbol{a} \geq \mathbf{0}, \boldsymbol{a} \neq \mathbf{0}$, $\bigwedge_{i=1}^{d} a^{(i)} Z^{(i)}$ has a regularly varying distribution tail of index α^0.

- In particular, hidden regular variation means both $Z^{(1)} \vee Z^{(2)}$ and $Z^{(1)} \wedge Z^{(2)}$ have regularly varying tail probabilities with indices α and α^0. Note

$$\eta = 1/\alpha^0 = \text{coefficient of tail dependence}$$

(Ledford and Tawn (1996,1997)).

- Define on $\aleph \cap \mathbb{E}_0$

$$S^0(\Lambda) = \nu^0\{\mathbf{x} \in \mathbb{E}^0 : |\mathbf{x}| \geq 1, \frac{\mathbf{x}}{|\mathbf{x}|} \in \Lambda\}$$

called the *hidden angular measure.*

Sub-model (cont)–Examples:

Example 1: $d = 2$; independent random quantities $B, \boldsymbol{Y}, \boldsymbol{U}$ with

$$P[B = 0] = P[B = 1] = 1/2$$

and $\boldsymbol{Y} = (Y^{(1)}, Y^{(1)})$ is iid with

$$P[Y^{(1)} > x] = x^{-1}, \quad x > 1$$

and

$$b(t) = t.$$

Let $\boldsymbol{U}$ have multivariate regularly varying distribution on $\mathbb{E}$ and no asymptotic independence and $\exists \alpha^0 > 1$, $b^0(t) \in RV_{1/\alpha^0}$, $\nu^0 \not\equiv 0$,

$$tP[\frac{\boldsymbol{U}}{b^0(t)} \in \cdot] \to \nu^0 \not\equiv 0.$$

Define

$$\boldsymbol{Z} = B\boldsymbol{Y} + (1 - B)\boldsymbol{U}$$

which has hidden regular variation, and the property

$$\boxed{S^0(\aleph^0) := \nu^0\{\mathbf{x} \in \mathbb{E}_0 : \|\mathbf{x}\| > 1\} < \infty.}$$

Example 2: For $d = 2$, define $\boldsymbol{Z} = (Z^{(1)}, Z^{(2)})$ iid and Pareto distributed with

$$P[Z^{(1)} > x] = x^{-1}, \quad x > 1, \; i = 1, 2.$$

Set

$$b(t) = t, \quad b^0(t) = \sqrt{t},$$

so that $b(t)/b^0(t) \to \infty$. Then $\boldsymbol{Z}$ is asymptotically independent and possesses hidden regular variation with

$$\nu^0\big([\mathbf{x}, \boldsymbol{\infty}]\big) = \big(x^{(1)} x^{(2)}\big)^{-1}.$$

On $\mathbb{E}$

$$tP[\frac{\boldsymbol{Z}}{b(t)} \in \cdot] \to^{v} \nu,$$

$\nu(\mathbb{E}_0) = 0$, and on $\mathbb{E}_0$

$$tP[\frac{\boldsymbol{Z}}{b^0(t)} \in \cdot] \to^{v} \nu^0,$$

and

$$\boxed{S^0(\aleph^0) := \nu^0\{\mathbf{x} \in \mathbb{E}_0 : \|\mathbf{x}\| > 1\} = \infty.}$$

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 45 of 71
Go Back
Full Screen
Close
Quit

HRV follows from

$$\begin{aligned} tP[Z^{(1)} > \sqrt{t}x, Z^{(2)} > \sqrt{t}y] & \\ &= \sqrt{t}P[Z^{(1)} > \sqrt{t}x]\sqrt{t}P[P[Z^{(2)} > \sqrt{t}y] \\ &\to \frac{1}{xy}. \end{aligned}$$

Example 3: Let

$$\boldsymbol{Z} = \Big(\frac{1}{U}, \frac{1}{1-U}\Big).$$

where $U \sim U(0,1)$. Then

$$\frac{1}{U} \bigwedge \frac{1}{1-U} \leq 2,$$

so $\frac{1}{U} \bigwedge \frac{1}{1-U}$ is not regularly varying.

Conclusions from Example 3:

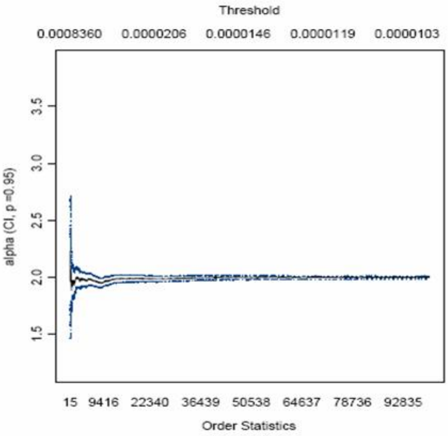
- $\boldsymbol{Z}$ possess asymptotic independence.
- $\boldsymbol{Z}$ does not possess hidden regular variation.
- Testing for asymptotic independence by testing the minimum of the components does not work here. Testing using the min of the components is a test for HRV, not asy indep.

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 46 of 71
Go Back
Full Screen
Close
Quit

9.3. Can We Detect Hidden Regular Variation?

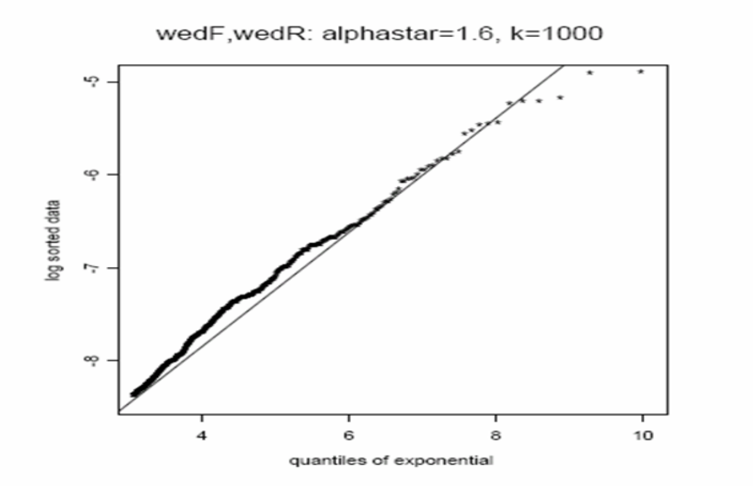
Example 1: Simulation.

5000 pairs of iid Pareto, $\alpha = 1$; $\alpha^0 = 2$. Hillplot for rank transformed data taking minima of components.



Example 2: UNC Wed (F,R).

QQ plot of rank transformed data using 1000 upper order statistics for UNC Wed (F,R); $\alpha = 1$ and $\hat{\alpha}^0 = 1.6$.



9.4. Estimating ν^0.

The hidden measure ν^0 has a spectral measure S^0 defined on $\aleph^0$, the unit sphere in $\mathbb{E}_0$:

$$S^0(\Lambda) := \nu^0\{\mathbf{x} \in \mathbb{E}_0 : \|\mathbf{x}\| > 1, \frac{\mathbf{x}}{\|\mathbf{x}\|} \in \Lambda\}.$$

S^0 may not necessarily be finite.

We estimate S^0 rather than ν^0.

Estimation procedure (Heffernan and Resnick, 2007) for estimating ν^0:

1. Replace the heavy tailed multivariate sample $\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_n$ by the n vectors of reciprocals of anti-ranks $1/\mathbf{r}_1, \dots, 1/\mathbf{r}_n$, where
$$r_i^{(j)} = \sum_{l=1}^{n} 1_{[Z_l^{(j)} \geq Z_i^{(j)}]}; \quad j = 1, \dots, d; \; i = 1, \dots, n.$$
2. Compute normalizing factors (min across components)
$$m_i = \bigwedge_{j=1}^{d} \frac{1}{r_i^{(j)}}; \quad i = 1, \dots, n,$$
and their order statistics
$$m_{(1)} \geq \cdots \geq m_{(n)}.$$
3. Compute the polar coordinates $\{(R_i, \boldsymbol{\Theta}_i); i = 1, \dots, n\}$ of
$$\{(1/r_i^{(j)}; j = 1, \dots, d); \; i = 1, \dots, n\}.$$
4. Estimate S^0 using the $\boldsymbol{\Theta}_i$ corresponding to $R_i \geq m_{(k)}$.

Details:

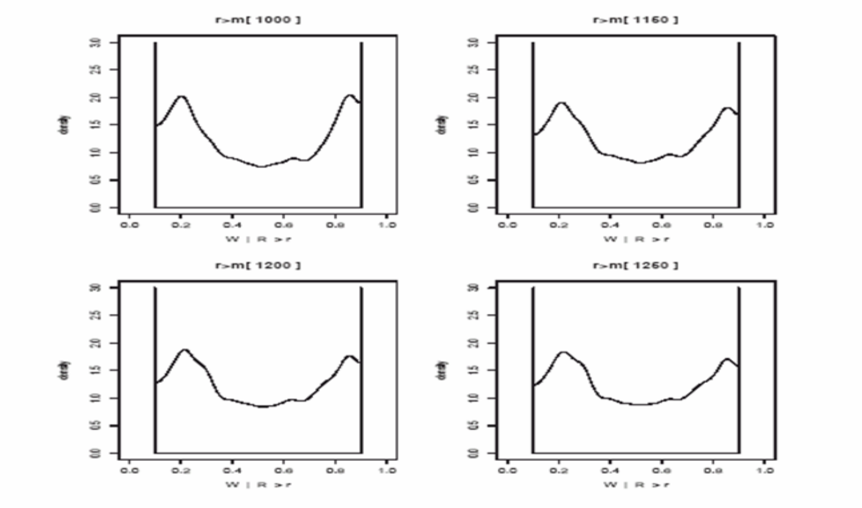
- If ν^0 is infinite, let $\aleph_0(K)$ be compact subset of $\aleph_0$.
  - For $d = 2$ where $\aleph$ can be parameterized as $\aleph = [0, \pi/2]$ and $\aleph_0 = (0, \pi/2)$, set $\aleph_0(K) = [\delta, \pi/2 - \delta]$ for some small $\delta > 0$.
- Then as $n \to \infty$ and $k = k(n) \to \infty$,, and $k/n \to 0$,

$$\frac{\sum_{i=1}^{n} 1_{[R_i \geq m_{(k)}, \boldsymbol{\Theta}_i \in \aleph_0(K)]} \epsilon_{\boldsymbol{\Theta}_i}}{\sum_{i=1}^{n} 1_{[R_i \geq m_{(k)}, \boldsymbol{\Theta}_i \in \aleph_0(K)]}} \Rightarrow S_0(\cdot \bigcap \aleph_0(K)).$$

- If ν^0 is finite, we can replace $\aleph_0(K)$ with $\aleph_0$. (But how would we know we can do this?)

Example.

UNC (F,R), April 26. Asymptotic independence present. Since S^0 may be infinite, we restricted estimation to the angular interval interval [0.1,0.9] instead of all of $[0, 1]$. All plots show the hidden measure to be bimodal with peaks around 0.2 and 0.85.



10. Conditional models.

Heffernan & Tawn models (Heffernan and Tawn, 2004):

$$P\Big[\frac{X-\beta(t)}{\alpha(t)} \le x | Y = t\Big] \to G(x), \quad t \to \infty. \tag{9}$$

Alternate approaches to asymptotic independence consider

$$P[X \le x | Y > t] \to G(x), \quad t \to \infty,$$

which comes from

$$tP[\big(X, \frac{Y}{t}\big) \in \cdot\,] \to H \times \nu_1$$

where H is a pm, $\nu_1(x,\infty] = x^{-1}$, $x > 1$. See Maulik et al. (2002).

With Jan Heffernan, meld 2 approaches (Heffernan and Resnick, 2007) and reformulate as

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \cdot\,\Big] \xrightarrow{v} \mu$$

where μ satisfies non-degeneracy assumptions. This relates to regular variation on the cone $\mathbb{E}_{\sqcap}$.

Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

10.1. Basic Convergence

Given a random vector (X, Y) with

$$F_Y(x) := P[Y \leq x] \in MDA(G_\gamma),$$

and $\exists\ b(\cdot) \in \mathbb{R}, a(\cdot) > 0$ such that for some $\gamma \in \mathbb{R}$, as $t \to \infty$,

$$\left(P\Big[\frac{Y - b(t)}{a(t)} \leq x\Big]\right)^t \to G_\gamma(x) = \exp\{-(1 + \gamma x)^{-1/\gamma}\}, \quad t \to \infty.$$

Further assume $\exists\ \beta(\cdot) \in \mathbb{R}, \alpha(\cdot) > 0$ and a Radon measure μ such that

$$tP\Big[\Big(\frac{X - \beta(t)}{\alpha(t)}, \frac{Y - b(t)}{a(t)}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot), \tag{10}$$

in $M_+([-\infty, \infty] \times (-\infty, \infty])$, and where μ is non-null and satisfies non-degeneracy conditions: for each fixed $y \in \{x : (1 + \gamma x)^{-1/\gamma} > 0\}$,

1. $\mu((-\infty, x] \times (y, \infty])$ is not a degenerate distribution function in x;
2. $\mu((-\infty, x] \times (y, \infty]) < \infty$.

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Page 54 of 71
Go Back
Full Screen
Close
Quit

10.2. Observations:

- The Basic Convergence (10) implies the conditioned limit

$$P\Big[\frac{X-\beta(t)}{\alpha(t)} \le x \Big| Y > b(t)\Big] \to \mu\big([-\infty, x] \times (0,\infty]\big),$$

  where the limit is assumed to be a proper probability distribution in x.

- WLOG can assume Y is heavy tailed and reduce the basic convergence to a more standard form:

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu^*(\cdot) \qquad (11)$$

  in $M_+([-\infty,\infty] \times (0,\infty])$ (μ^* is modified from μ). For instance,

$$Y^* = b^{\leftarrow}(Y) \quad \text{and} \quad b(t) = \Big(\frac{1}{1-F_Y}\Big)^{\leftarrow}(t).$$

- Suppose (X, Y^*) are regularly varying on $[0,\infty]^2 \setminus \{\mathbf{0}\}$.
  - With no asymptotic independence in the EVT sense, Basic Convergence automatically holds.
  - With asymptotic independence in EVT sense, Basic Convergence is an extra assumption.

10.3. Other Consequences.

- A convergence to types argument implies variation properties of $\alpha(\cdot)$ and $\beta(\cdot)$: Suppose (X, Y^*) satisfy the condition (11). $\exists$ two functions $\psi_1(\cdot), \psi_2(\cdot)$, such that for all $c > 0$,

$$\lim_{t\to\infty} \frac{\alpha(tc)}{\alpha(t)} = \psi_1(c), \quad \lim_{t\to\infty} \frac{\beta(tc) - \beta(t)}{\alpha(t)} \to \psi_2(c). \tag{12}$$

locally uniformly.

- This means

$$\alpha(\cdot) \in RV_\rho, \; \rho \in \mathbb{R} \quad \text{and} \quad \psi_1(c) = c^\rho, \; c > 0.$$

- $\exists$ important cases where $\psi_2 \equiv 0$ (bivariate normal). However, if $\psi_2 \not\equiv 0$, then ((Geluk and de Haan, 1987, page 16), Bingham et al. (1987))

$$\psi_2(x) = \begin{cases} k\frac{(x^\rho - 1)}{\rho}, & \text{if } \rho \neq 0, \; x > 0, \\ k \log x, & \text{if } \rho = 0, \; x > 0, \end{cases} \tag{13}$$

for $k \neq 0$.

CORNELL
Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶

Go Back
Full Screen
Close
Quit

10.4. When can both components in the basic convergence be standardized?

- Can sometimes also standardize the X variable so that

$$
\begin{aligned}
tP\Big[\frac{\beta^{\leftarrow}(X)}{t} \le x, \frac{Y^*}{t} > y\Big] = tP\Big[\frac{X^*}{t} \le x, \frac{Y^*}{t} > y\Big]
&\to \mu^*\big([-\infty, \psi_2(x)] \times (y,\infty]\big)\\
&= \mu^{**}\big([-\infty, x] \times (y,\infty]\big) \qquad (t \to \infty). \qquad (14)
\end{aligned}
$$

When?? Short version: When and only when μ^* (or μ) is not a product measure (Das and Resnick, 2008).

  - When is μ^* a product measure?
    Answer: $\mu^* = H \times \nu_1$ iff $\psi_1 \equiv 1$ ($\alpha(\cdot)$ is sv) and $\psi_2 \equiv 0$.
  - If you can standardize, how do you do it?
    - One answer: If $\beta(t) \ge 0$ and $\beta^{\leftarrow}$ is non-decreasing on the range of X, then (14) is possible (provided μ is NOT a product).
    - If the condition $\beta(t) \ge 0$ and $\beta^{\leftarrow}$ is non-decreasing fails, then a transformation of X allows one to reduce the problem to the previous case.

- If we have $X \geq 0$ and both regular variation on $[0,\infty]^2 \setminus \{\mathbf{0}\}$

$$tP\Big[\Big(\frac{X}{a'(t)}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} \nu_*(\cdot)$$

  and (14):

$$tP\Big[\frac{\beta^{\leftarrow}(X)}{t} \leq x, \frac{Y^*}{t} > y\Big] \to \mu^*\big([-\infty, \psi_2(x)] \times (y, \infty]\big)$$

  on $[0,\infty] \times (0,\infty]$, then we have a form of hidden regular variation since

$$[0,\infty] \times (0,\infty] \subset [0,\infty]^2 \setminus \{\mathbf{0}\}.$$

More on this in a bit.

10.5. Form of the limit.

10.6. Case 1. Assume μ is not a product.

Then can standardize X and for the case that $\beta(t) \geq 0$ and $\beta(t) \uparrow$

$$tP\Big[\frac{\beta^{\leftarrow}(X)}{t} \leq x, \frac{Y^*}{t} > y\Big] = tP\Big[\frac{X^*}{t} \leq x, \frac{Y^*}{t} > y\Big]$$
$$\to \mu^*\big([0, \psi_2(x)] \times (y, \infty]\big) = \mu^{**}([0, x] \times (y, \infty]), \quad (t \to \infty)$$

on $[0, \infty] \times (0, \infty]$. This is standard regular variation on the cone $[0, \infty] \times (0, \infty]$ so

$$\mu^{**}(c\Lambda) = c^{-1}\mu^{**}(\Lambda).$$

$\exists$ angular measure: Let

$$\|(x, y)\| = x + y, \quad \aleph = \{(w, 1 - w) : 0 \leq w < 1\}$$

and

$$\mu^{**}\{\mathbf{x} : \|\mathbf{x}\| > r, \frac{\mathbf{x}}{\|\mathbf{x}\|} \in A\} = r^{-1}S(A),$$

where S is a measure on $\aleph$ or $[0, 1)$.

Warning:

S does not have to be finite

but to guarantee

$$P[\frac{X^*}{t} \le x | Y^* > t] \to H^{**}(x) = \mu^{**}([0,x] \times (1,\infty])$$

is proper pm, need

$$\int_{[0,1)} (1-w)S(dw) = 1. \tag{15}$$

Conclusions for Case 1:

- Can write $\mu^{**}([0,x] \times (y,\infty])$ as function of S and get characterization of the class of limit measures.
- Hence, limit measures μ^{**} are in 1-1 correspondence with class of measures on $[0,1)$ satisfying (15).
- Alternatively, a scaling argument gives class of μ^{**} with following description:

$$\mu^{**}([0,x] \times (y,\infty]) = y^{-1} H^{**}(\frac{x}{y}), \quad (x,y) \in [0,\infty] \times (0,\infty]$$

for any proper prob distribution H^{**}.

10.7. Case 2: Assume μ is a product.

Any measure of the form

$$\mu^{**}([0,x]\times(y,\infty]) = y^{-1}H^{**}(x),$$

for a prob distribution H^{**} is a possible limit.

End of story for Case 2.

10.8. Random norming and change of coordinates.

Suppose μ is not a product. Then both variables X, Y can be standardized

$$(X,Y)\mapsto(X^*,Y^*)$$

and taking the ratio X^*/Y^* gives a product form limit. After transforming

$$(X^*,Y^*)\mapsto\Big(\frac{X^*}{Y^*},Y^*\Big)$$

we get

$$tP\Big[\Big(\frac{X^*}{Y^*},\frac{Y^*}{t}\Big)\in\cdot\Big]\to G\times\nu_1(\cdot)$$

in $M_+([0,\infty]\times(0,\infty])$ where

$$\nu_1(y,\infty] = y^{-1}, \quad G(x) = \int_{[0,\frac{x}{1+x}]} (1-w)S(dw).$$

(Heffernan and Resnick, 2007, Maulik et al., 2002).

The condition

$$tP\Big[\Big(\frac{X^*}{Y^*},\frac{Y^*}{t}\Big)\in\cdot\Big]\to G\times\nu_1(\cdot) \tag{16}$$

can be detected by reduction to one dimensional regular variation: We have (16) iff

$$Y^* 1_{[X^*/Y^*\in\Lambda]}$$

has standard regularly varying distribution for any $\Lambda\in\mathcal{B}(\mathbb{R}_+)$, or iff

$$aX^*\wedge Y^*$$

has a standard regularly varying distribution for any $a>0$. (Das and Resnick, 2008)

Statistical implications?? (Das & Resnick, 200∞.)

10.9. The cone $\mathbb{E}_{\sqcap} = [0,\infty] \times (0,\infty]$ and conditioned limit laws.

10.9.1. Limit measure is a product.

- Suppose (X^*, Y^*) is standard regularly varying on the cone $\mathbb{E}_{\sqcap}$, such that

$$t\mathbb{P}\left[\left(\frac{X^*}{t}, \frac{Y^*}{t}\right) \in \cdot\right] \xrightarrow{v} \mu^{**}(\cdot) \qquad \text{in } M_+(\mathbb{E}_{\sqcap}) \tag{17}$$

  for some Radon measure $\mu^{**}(\cdot)$ on $\mathbb{E}_{\sqcap}$ satisfying the non-degeneracy conditions. Then $\mu^{**}(\cdot)$ cannot be a product measure.

- Suppose we have

$$t\mathbb{P}\left[\left(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y^*}{t}\right) \in \cdot\right] \xrightarrow{v} G \times \nu_1(\cdot) \text{ in } M_+([-\infty,\infty] \times (0,\infty]). \tag{18}$$

  Then we cannot have a monotone standardization function $f(\cdot)$ such that

$$t\mathbb{P}\left[\left(\frac{f(X)}{t}, \frac{Y^*}{t}\right) \in \cdot\right] \xrightarrow{v} \mu(\cdot) \qquad \text{in } M_+(\mathbb{E}_{\sqcap}). \tag{19}$$

  No standardization is possible. (Das and Resnick, 2008).

10.9.2. Limit measure is not a product.

Given the basic convergence,

$$tP\Big[\Big(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y-b(t)}{a(t)}\Big) \in \cdot\Big] \xrightarrow{v} \mu(\cdot), \tag{20}$$

where μ is NOT a product measure, can always standardize so that

$$P\Big[\Big(\frac{X^*}{t}, \frac{Y^*}{t}\Big) \in \cdot\Big] \xrightarrow{v} \mu^{**}(\cdot), \qquad \text{in } M_+(\mathbb{E}_\sqcap). \tag{21}$$

And vice versa. Given (21), $\exists$ monotone transformations f_1, f_2 such that

$$(X, Y) = \big(f_1(X^*), f_2(Y^*)\big)$$

satisfies (20).

Regular Variation
Regular variation on cones
Standardization
1-dim reduction
Asy indep EVT
Examples
How standardize?
Data examples
Hidden reg var (HRV)
Conditional models
Consistency
Title Page
◀◀ ▶▶
◀ ▶
Go Back
Full Screen
Close
Quit

11. Consistency

- In practice, should one condition on X or Y?
- What if one could do either?

Then the distribution is in the domain of attraction of an EV distribution.

Theorem. Suppose we have a bivariate random vector $(X, Y) \in \mathbb{R}^2$ and functions $\alpha(\cdot), a(\cdot), \chi(\cdot), c(\cdot) > 0$ and $\beta(\cdot), b(\cdot), \phi(\cdot), d(\cdot) \in \mathbb{R}$ such that

$$t\mathbb{P}\left[\left(\frac{X-\beta(t)}{\alpha(t)}, \frac{Y-b(t)}{a(t)}\right) \in \cdot\right] \xrightarrow{v} \mu(\cdot) \text{ in } M_+([-\infty,\infty] \times (-\frac{1}{\gamma},\infty]), \tag{22}$$

$$t\mathbb{P}\left[\left(\frac{X-\phi(t)}{\chi(t)}, \frac{Y-d(t)}{c(t)}\right) \in \cdot\right] \xrightarrow{v} \nu(\cdot) \text{ in } M_+((-\frac{1}{\lambda},\infty] \times [-\infty,\infty]) \tag{23}$$

for some $\lambda, \gamma \in \mathbb{R}$.

Then (X, Y) is in the domain of attraction of a multivariate extreme value distribution on

$$\mathbb{E}_{\lambda,\gamma} := [-\frac{1}{\lambda}, \infty] \times [-\frac{1}{\gamma}, \infty] \setminus \{(-\frac{1}{\gamma}, -\frac{1}{\lambda})\},$$

that is,

$$t\mathbb{P}\left[\left(\frac{X - \beta(t)}{\alpha(t)}, \frac{Y - b(t)}{a(t)}\right) \in \cdot\right] \xrightarrow{v} (\mu \diamond \nu)(\cdot) \text{ in } M_+(\mathbb{E}_{\lambda,\gamma})$$

where $(\mu \diamond \nu)(\cdot)$ is a Radon measure on $M_+(\mathbb{E}_{\lambda,\gamma})$.

Conclusion: So the conditioned limit theory is only different than classical EVT if we assume we can condition only on one variable but not on the other.

Final thoughts

- How practical is all this?
- When should you try to use this theory rather than EVT?
- Can this theory be used to supplement EVT in the presence of asymptotic independence? This seems the most likely road to applicability.
- How do we do estimation and inference?

We are thinking about all this.

Contents

References

N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation.* Cambridge University Press, 1987.

B. Das and S. Resnick. Conditioning on an extreme component: Model consistency and regular variation on cones. Technical report, Cornell University, School of ORIE, 2008. In preparation.

Laurens de Haan. A characterization of multidimensional extreme-value distributions. *Sankhyā Ser. A*, 40(1):85–88, 1978. ISSN 0581-572X.

L. de Haan and A. Ferreira. *Extreme Value Theory: An Introduction.* Springer-Verlag, New York, 2006.

L. de Haan and S. I. Resnick. Limit theory for multivariate sample extremes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 40(4): 317–337, 1977.

J. L. Geluk and L. de Haan. *Regular Variation, Extensions and Tauberian Theorems*, volume 40 of *CWI Tract*. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1987. ISBN 90-6196-324-9.

J.E. Heffernan and S.I. Resnick. Hidden regular variation and the rank transform. *Adv. Appl. Prob.*, 37(2):393–414, 2005.

Title Page
◀◀ ▶▶
◀ ▶
Page 70 of 71
Go Back
Full Screen
Close
Quit

J.E. Heffernan and S. Resnick. Limit laws for random vectors with an extreme component. *Ann. Appl. Probab.*, 17(2):537–571, 2007. ISSN 1050-5164. doi: 10.1214/105051606000000835.

J.E. Heffernan and J.A. Tawn. A conditional approach for multivariate extreme values (with discussion). *JRSS B*, 66(3):497–546, 2004.

C. Klüppelberg and S. Resnick. The Pareto copula, aggregation of risks and the emperor's socks. *Journal of Applied Probability*, 45(1): 67–84, 2008.

K. Maulik, S.I. Resnick, and H. Rootzén. Asymptotic independence and a network traffic model. *J. Appl. Probab.*, 39(4):671–699, 2002. ISSN 0021-9002.

T. Mikosch. How to model multivariate extremes if one must? *Statist. Neerlandica*, 59(3):324–338, 2005. ISSN 0039-0402.

T. Mikosch. Copulas: Tales and facts. *Extremes*, 9(1):3–20, 2006. ISSN 1386-1999 (Print) 1572-915X (Online).

S.I. Resnick. *Extreme Values, Regular Variation and Point Processes.* Springer-Verlag, New York, 1987.

S. Resnick. *Heavy Tail Phenomena: Probabilistic and Statistical Modeling.* Springer Series in Operations Research and Financial Engineering. Springer-Verlag, New York, 2007. ISBN: 0-387-24272-4.