2nd Conference in Actuarial Science & Finance in Samos, September 20-22, 2002

Portfolio optimization models with transaction costs and initial holdings

Marius Rădulescu and Sorin Rădulescu
Institute of Mathematical Statistics and Applied Mathematics, Casa Academiei Romane, Calea13 Septembrie nr.13, Bucharest 5, RO-76100, ROMANIA,
email: mradulescu@csm.ro

Constanța Zoie Rădulescu
Research Institute for Informatics, 8-10 Averescu Avenue, 71316 Bucharest 1, ROMANIA, e-mail: radulescu@u3.ici.ro,

Abstract. In this paper we extend Markowitz's portfolio selection model to include transaction costs in the presence of initial holdings for the investor. We present also a chance constrained model and a safety first model which also include transaction costs in the presence of initial holdings for the investor.Our approach is new. At period $t = 0$, the investor purchases and sells assets in order to obtain a fixed sum which he will put in his pocket. Our aim is to obtain an optimal portfolio which has a minimum risk or a maximum return. The portfolio selection models for the case of transaction costs are solved by computer simulation. Our portfolio optimization module (PORTOPT) computes the optimal portfolio during the horizon period by solving a quadratic programming problem with linear and complementarity restrictions. Our optimization model is integrated in a risk management decision support system.

1. Introduction

The standard portfolio model is one where an investor allocates his initial wealth among *n* risky assets. Assuming the assets returns are stochastic, Harry Markowitz (the 1990 Nobel laureate) in his pioneering work [7] described a theory postulating that rational investors should select a portfolio from the set of all "feasible" portfolios which offers minimum risk for a given level of expected return and maximum expected return for a given level of risk. Such a portfolio is said to be efficient. The objective of the portfolio selection problem is then to determine the set of efficient portfolios.

Risk is measured in terms of the variance of the total return. The above model of Markowitz is known as the mean-variance model.

Since its introduction in the fifties, the model has dominated a great deal of the literature in portfolio analysis. For references concerning portfolio theory see: [4,8,9,11,12].

The classical Markowitz's portfolio selection model is a single period model which does not suppose the existence of initial holdings for the investor and does not takes into account the transaction costs. It is static and assumes a frictionless market. Pogue [10] had extended Markowitz's portfolio selection model to include transaction costs, short sales, leverage policies and taxes.

The mathematical problem of optimally managing a portfolio of securities when there are transaction costs has received considerable research attention in recent years.

For the classical problem where the objective is to maximize expected utility of terminal wealth, most of the attention has been devoted to the case of proportional transaction costs, that is, to the case where the cost associated with a transaction is proportional to the amount of money that is shifted between the securities. Representative of work in this category are the papers [1,2,3,5,6,13,14].

In this paper we intend to extend Markowitz's portfolio selection model to include transaction costs in the presence of initial holdings for the investor. Our approach is new.

In the following we shall suppose that:

- At period $t = 0$ the initial holdings of the investor are described by the vector $b = (b_1, b_2, ..., b_n)$, that is the investor had purchased before period $t = 0$ a quantity b_i of asset i, i=1,2.....,n.
- The transaction costs for the assets are nonlinear maps (usually they are piecewise linear) of the volume of assets purchased or sold. We shall denote by c_1 the fee paid by the investor for purchasing some assets. We shall denote by c_2 the fee paid by the investor for selling some assets. This means that if the investor has a sum M of money and if he purchases assets then he will pay as a commission a sum c_1 (M). If the investor sells some of his assets of value M then he will pay a commission of c_2 (M). Denote by $c_3(M)$ the sum of money the investor must pay for purchasing assets with net price M. One can easily see that $c_3(M) - c_1(c_3(M)) = M$.

  Note that $0 = c_3(0) = c_i(0) \le c_i(s) \le s \le c_3(s)$ for every $s \ge 0,\ i \in \{1,2\}$ and all the

  maps c_1, c_2, c_3 are nondecreasing.
- A unit of time was chosen so that the rates of return for the n assets where the investor wants to make the investment can be computed. We shall denote by ξ_i the rate of return for the asset i, i=1,2,...,n. Obviously that ξ_i are random variables. We shall denote by μ_i the expected value of ξ_i. Let $\xi=(\xi_1,\xi_2,....,\xi_n)$ be the random vector of the rates of return for the n assets and $\mu=(\mu_1, \mu_2,....,\mu_n)$ the vector of means (expected rates of return).

At period $t = 0$, the investor wants to make a revision of his portfolio, that is he purchases and sells assets in order to obtain a sum W_0. If W_0 is positive the investor will put it in his pocket (that is the investor will use the money for his personal consumption). If W_0 is negative the investor will use the sum $|W_0| = -W_0$ from his own pocket to purchase assets. To do this the investor will purchase a portfolio of assets $u=(u_1,u_2,...,u_n)$ and will sell a portfolio of assets $v=(v_1,v_2,...,v_n)$. Let $x=b+u-v$ be the investor's portfolio at moment $t = 0$ after the revision. To purchase the portfolio u the investor will need a sum of money equal to $c_3\left(\sum_{i=1}^{n} u_i\right)$. When the investor sells the portfolio v, he will obtain a sum of money equal to

$$\sum_{i=1}^{n} v_i - c_2\left(\sum_{i=1}^{n} v_i\right). \tag{1}$$

Consequently we can write the equality:

$$W_0 = \sum_{i=1}^{n} v_i - c_2\left(\sum_{i=1}^{n} v_i\right) - c_3\left(\sum_{i=1}^{n} u_i\right). \tag{2}$$

Obviously we must have

$$b_i + u_i - v_i \ge 0,\ i=1,2,...n. \tag{3}$$

At period $t = 1$ the investor's return will be:

$$\xi^T x = \sum_{i=1}^{n} \xi_i x_i = \sum_{i=1}^{n} \xi_i (b_i + u_i - v_i) \tag{4}$$

The expected value of portfolio return will be:

$$\mathrm{E}(\xi^T x) = \sum_{i=1}^{n} \mathrm{E}(\xi_i) x_i = \sum_{i=1}^{n} \mu_i x_i = \sum_{i=1}^{n} \mu_i (b_i + u_i - v_i) \tag{5}$$

In the portfolio selection models we are going to study we shall suppose that the investor is rational, that is at period $t = 0$, when the investor is rebalancing his portfolio consisting of his initial holdings, he will not purchase assets which he will sell at the same moment.

In this case we can write that:

$$u_i v_i = 0, \quad \text{for every} \quad i \in \{1,2,\ldots,n\} \tag{6}$$

Since $u_i \geq 0,\ v_i \geq 0$, for every $i \in \{1,2,\ldots,n\}$ it follows that the condition (6) may be written in vectorial form as:

$$u^T v = 0 \tag{7}$$

We shall call the condition (7) the investor's rationality condition.

2. A risk minimization model

We shall define the investment risk as the variance of the random variable return. Let $c_{ij} = E((\xi_i - \mu_i)(\xi_j - \mu_j))$, $1 \leq i, j \leq n$ and $C=(c_{ij})$. Note that c_{ij} represents the covariance of the random variables ξ_i and ξ_j. The matrix C is the covariance matrix for the random vector ξ.

One can easily see that the variance of the return is equal to:

$$D^2(\xi^T x) = E\left((\xi^T x - \mu^T x)^T (\xi^T x - \mu^T x)\right) =$$
$$= E\left(x^T (\xi - \mu)(\xi - \mu)^T x\right) = x^T E\left((\xi - \mu)(\xi - \mu)^T\right) x = x^T C x$$

In this model the investor seeks to:

- withdraw for consumption at period $t = 0$ a sum of money equal to W_0 from his portfolio. To do this he will rebalance his portfolio by purchasing and selling assets of his portfolio (initial holdings).
- obtain with minimum risk, at period $t = 1$ an expected net return at least equal to W_1.

The equations for the risk minimization model are:

$$\begin{cases} \min\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(b_i + u_i - v_i)(b_j + u_j - v_j)\right) \\ W_0 = \sum_{i=1}^{n} v_i - c_2\left(\sum_{i=1}^{n} v_i\right) - c_3\left(\sum_{i=1}^{n} u_i\right) \\ \sum_{i=1}^{n} \mu_i (b_i + u_i - v_i) \geq W_1 \\ b_i + u_i - v_i \geq 0, \quad i = 1,2,\ldots,n \\ u_i v_i = 0, \quad i = 1,2,\ldots,n \\ v_i \geq 0,\ u_i \geq 0, \quad i = 1,2,\ldots,n \end{cases} \tag{8}$$

Let $e = (1,1,\ldots,1)^T$ be the n dimensional vector having all coordinates equal to 1.

In the vectorial form the model can be written as follows:

$$\begin{cases} \min\left[(b+u-v)^T C(b+u-v)\right] \\ e^T v - c_2\left(e^T v\right) - c_3\left(e^T u\right) = W_0 \\ \mu^T(b+u-v) \geq W_1 \\ u^T v = 0,\ b+u-v \geq 0,\ u \geq 0,\ v \geq 0 \end{cases} \tag{9}$$

Note that the input data in the model is represented by:

- the covariance matrix of the rates of return $C=(c_{ij})$
- the vector of the mean (expected) rate of return $\mu=(\mu_1, \mu_2, \ldots, \mu_n)$
- the vector $b=(b_1, b_2, \ldots, b_n)$, of investor's initial holdings, that is the investor portfolio at period $t = 0$.

The user's parameters are represented by the numbers:

W_0 = the sum the investor wish to withdraw for consumption at period $t = 0$.
W_1 = the inferior limit for the return the investor wish to obtain at period $t = 1$. (Obviously W_1 does not contain W_0 in its definition).

The decision variables are represented by the portfolios u and v the investor purchases and sells. The purpose of the model is to find the vectors u and v, which minimize the objective map.

One obtains a quadratic programming problem. We note that the set of feasible solutions is not defined by linear maps and therefore it is not convex. This is due to the complementarity restrictions (7). One can easily see that the set of feasible solutions is the union of 2^n convex sets.

In order to compute the exact solution of problem (9) in the case of linear transaction costs, an exhaustive combinatorial search can be performed.

We can solve a quadratic programming problem with linear restrictions for each convex component of the set of feasible solutions. Because of the computational effort this approach becomes difficult for n larger than 10 and certainly intractable for n larger than 25. The exhaustive search can be improved by using branch and bound methods, which can reduce the required computational effort. These methods require a procedure for computing upper and lower bounds for the objective map.

Another approach for solving the problem (9) is by using computer simulation.

An important problem for the investor is to find the restrictions he might take into account when he chooses the parameters W_0 and W_1. Denote

$$c_4(s) = s - c_2(s),\ s \geq 0\,.$$

If the map c_4 is nondecreasing then the parameter W_0 can be chosen everywhere in the interval $\left(-\infty, c_4\left(e^{\mathrm{T}} b\right)\right)$.

Denote

$$M = \left\{(u, v) \in \mathbf{R}^{2n} : u^T v = 0,\ b + u - v \geq 0,\ u \geq 0,\ v \geq 0\right\}$$

$$Q(W_0) = \left\{(u, v) \in M : e^T v - c_2\left(e^T v\right) - c_3\left(e^T u\right) = W_0\right\},\ L(u, v) = \mu^T(b + u - v)$$

$$W_1' = \min\left\{L(u,v) : (u,v) \in Q(W_0)\right\},\ W_1'' = \max\left\{L(u,v) : (u,v) \in Q(W_0)\right\}$$

The investor can choose the parameter W_1 everywhere in the closed interval $\left[W_1', W_1''\right]$.

3. The maximization of the average return model

In the framework of this model the investor seeks to purchase and sell assets so that:

- he will maximize his average net return at the moment $t = 1$ keeping the risk under a prescribed level r.
- he will withdraw for consumption a sum W_0 at period $t = 0$.

The equations for the maximization of the average return model are:

$$\begin{cases} \max\left[\mu^T (b + u - v)\right] \\ e^T v - c_2(e^T v) - c_3(e^T u) = W_0 \\ (b + u - v)^T C(b + u - v) \le r \\ u^T v = 0 \\ b + u - v \ge 0,\ u \ge 0,\ v \ge 0 \end{cases} \tag{10}$$

The investor can choose the parameter W_0 taking into account the same restrictions as in the preceding model. Denote

$r_1 = \min\left\{(b + u - v)^T C(b + u - v) : (u, v) \in Q(W_0)\right\}$ and

$r_2 = \max\left\{(b + u - v)^T C(b + u - v) : (u, v) \in Q(W_0)\right\}$.

Then the investor can choose the parameter r everywhere in the closed interval $[r_1, r_2]$.

4. The mean-variance model

This model studies the case of an investor who is characterized by a risk aversion coefficient. We shall denote the risk aversion coefficient by θ and we shall suppose that $\theta \in [0,1]$. The case when θ takes small values, that is near 0, corresponds to the case the investor has a great aversion towards risk, that is safety is essential for the investor and the amount of the return is not so important. The case when θ takes great values, that is near 1, corresponds to the case of a risk lover investor, that is the amount of the return is essential and the safety is not so important.

The objective map of the model is a linear combination between the risk and the expected value of the portfolio return. It depends on the investor's risk aversion coefficient θ. The aim of the investor is to minimize the objective map and to withdraw for consumption at period $t = 0$ a sum W_0.

The equations for the mean-variance model model are:

$$\begin{cases} \min\left[(1 - \theta)(b + u - v)^T C(b + u - v) - \theta\mu^T (b + u - v)\right] \\ e^T v - c_2(e^T v) - c_3(e^T u) = W_0 \\ u^T v = 0 \\ b + u - v \ge 0,\ u \ge 0,\ v \ge 0 \end{cases} \tag{11}$$

In the following we shall present two single period portfolio selection models. One of them is a safety first model and the other one is a chance constrained model.

5. A safety first model

In the framework of this model the investor seeks to purchase and sell assets so that:

- he will minimize the probability that his net return at the moment $t = 1$ falls below a threshold W_1. The above mentioned event is considered to have some bad consequences for the investor.
- he will withdraw for consumption a sum W_0 at period $t = 0$.

The equations for the safety first model are:

$$\begin{cases} \min \mathrm{P}\left(\xi^{\mathrm{T}}(b+u-v) < W_1\right) \\ \mu^T(b+u-v) \geq W_2 \\ e^T v - c_2\left(e^T v\right) - c_3\left(e^T u\right) = W_0 \\ u^T v = 0, \quad b+u-v \geq 0, \quad u \geq 0, \quad v \geq 0 \end{cases} \tag{12}$$

In the above model the decision variables (the unknowns) are the vectors u and v.
In case the random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ has a multivariate normal distribution with the covariance matrix C and vector of means μ one can easily see that

$$\mathrm{P}\left(\xi^{\mathrm{T}}(b+u-v) < W_1\right) = \phi\left(\frac{W_1 - \mu^{\mathrm{T}}(b+u-v)}{\sqrt{(b+u-v)^T C\,(b+u-v)}}\right)$$

Here ϕ is the standard normal distribution that is:

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{s^2}{2}\right) \mathrm{d}s \tag{13}$$

Denote $Q = \{(u, v) \in \mathbf{R}^{2n} : u$ and v verify all the restrictions from the model $(12)\}$
If $(b+u-v)^T C\,(b+u-v) > 0$ for all $(u, v) \in Q$ then the above model can be written as follows:

$$\begin{cases} \min\left[\dfrac{W_1 - \mu^{\mathrm{T}}(b+u-v)}{\sqrt{(b+u-v)^T C\,(b+u-v)}}\right] \\ \mu^T(b+u-v) \geq W_2 \\ e^T v - c_2\left(e^T v\right) - c_3\left(e^T u\right) = W_0 \\ u^T v = 0, \quad b+u-v \geq 0, \quad u \geq 0, \quad v \geq 0 \end{cases} \tag{14}$$

Note that the input data in the above model is represented by:

- the covariance matrix of the rates of return $C = (c_{ij})$
- the vector of the mean (expected) rate of return $\mu=(\mu_1, \mu_2, \ldots, \mu_n)$
- the vector $b=(b_1,b_2,\ldots,b_n)$, of investor's initial holdings, that is the investor portfolio at period $t = 0$.

The user's parameters are represented by the numbers:

- W_0 = the sum the investor wish to withdraw for consumption at period $t = 0$.
- W_1 = the inferior limit for the return the investor wish to obtain at period $t = 1$. (Obviously W_1 does not contain W_0 in its definition).
- W_2 = the inferior limit for the expected net return the investor wish to obtain at period $t = 1$.

The decision variables are represented by the portfolios u and v the investor purchases and sells. The purpose of the model is to find the vectors u and v, which minimize the objective map.

One obtains a fractional programming problem which can be reduced by a standard technique to a quadratic programming problem. One obtains a fractional programming problem which can be reduced by a standard technique to a quadratic programming problem. Unfortunately even in the case the transaction maps are linear, the set of feasible solutions is not convex.

Since in the general model, that is in the stochastic programming model (12), we have several nonlinearities is necessary to use computer simulation to solve it.

6. A chance constrained model

In the framework of this model the investor seeks to purchase and sell assets so that:

- he will maximize his average return at the moment $t = 1$ keeping the probability for obtaining a return under some prescribed sum W_1 less than a prescribed small number $p \in (0,1)$.
- he will withdraw for consumption a sum W_0 at period $t = 0$.

The equations for the chance constrained model are:

$$\begin{cases} \max\left[\mu^T (b + u - v)\right] \\ e^T v - c_2(e^T v) - c_3(e^T u) = W_0 \\ \mathrm{P}\left(\xi^T (b + u - v) < W_1\right) \le p \\ u^T v = 0, \quad b + u - v \ge 0, \quad u \ge 0, \quad v \ge 0 \end{cases} \tag{15}$$

In case the random vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ has a multivariate normal distribution with the covariance matrix C and vector of means μ then the above model can be written as follows:

$$\begin{cases} \max\left[\mu^T (b + u - v)\right] \\ \dfrac{W_1 - \mu^T (b + u - v)}{\sqrt{(b + u - v)^T C \,(b + u - v)}} \le \phi^{-1}(p) \\ e^T v - c_2(e^T v) - c_3(e^T u) = W_0 \\ u^T v = 0, \quad b + u - v \ge 0, \quad u \ge 0, \quad v \ge 0 \end{cases} \tag{16}$$

Of course the above model is correct only if we suppose that
$(b + u - v)^T C \,(b + u - v) > 0$ for all vectors u and v that verify the last two sets of restrictions.

Since the above models, in the general case are very complex, it is necessary to use computer simulation in order to solve them.

Monte Carlo simulation is the idea of using statistical trials to get an approximate solution to a problem. We use Monte Carlo simulation to solve the optimization portfolio selection problems of the above mentioned type and some multiperiod portfolio models. A number of simulations are carried out using the observed changes in the market factors over the last N periods to generate N hypothetical portfolio profits. Some other simulations are carried out choosing a statistical distribution that is believed to adequately capture or approximate the possible changes in the market factors. Then, a pseudo-random number generator is used to generate thousands or perhaps tens of thousands of hypothetical changes in the market factors. These are then used to construct thousands of hypothetical portfolio profits and losses on the current portfolio, and the distribution of possible portfolio profit or loss. From this distribution one can determine also the value at risk of the portfolio.

The experiments using Monte Carlo simulation aim to obtain an optimal portfolio, which has a minimum risk or a maximum return.

We built a portfolio simulation module (PORTOPT) that simulates the income produced by the portfolio during the horizon period and computes market value at the end of the horizon. Total return, capital gains and losses and income are all computed.

Our simulation module is integrated in a risk management decision support system.

7. Numerical results

We consider the risk minimization model in the case of proportional transaction costs. We denote by q_1 (respectively q_2) the transaction cost rate for purchasing (respectively selling) assets. That is $c_1(s) = q_1 s$, $c_2(s) = q_2 s$. Then $c_3(s) = \dfrac{s}{1 - q_1} = q_3 s$. We study the cases: q_1=0.05, q_2=0.06 and q_1=0.08, q_2=0.09. We note that the increase of the transaction costs implies the decrease of the transaction volumes.

Note that the upper limit for W_0 =18142, $W_1^{'}$ =-14209 and $W_1^{''}$ =235. In tables 1 and 2 are presented the assets purchased (vector u) and sold (vector v) in the case q_1=0.05, q_2=0.06. In tables 3 and 4 are presented the assets purchased (vector u) and sold (vector v) in the case q_1=0.08, q_2=0.09.

Table 1

Assets	Vector b	Vector u	Vector v	Vector x
A1	500	0	0	500
A2	1500	0	4.191	1495.808
A3	1200	0	13.094	1186.905
A4	2026	1004.198	0	3030.198
A5	2500	0	0	2500
A6	2100	0	1107.236	992.763
A7	3500	0	0	3500
A8	2800	0	0	2800
A9	1700	0	0	1700
A10	1300	0	0	1300
SUM	19126	1004.198	1124.522	

Table 2

W0	0
W1	150
*q*1	0.05
*q*2	0.06
*q*3	1.052631579

Table 3

Assets	Vector *b*	Vector *u*	Vector *v*	Vector *x*
A1	500	0	0	500
A2	1500	0	0	1500
A3	1200	0	27.065	1172.934
A4	2026	597.384	0	2623.384
A5	2500	0	0	2500
A6	2100	0	686.485	1413.514
A7	3500	0	0	3500
A8	2800	0	0	2800
A9	1700	0	0	1700
A10	1300	0	0	1300
SUM	19126	597.3849	713.5510	

Table 4

W0	0
W1	150
*q*1	0.08
*q*2	0.09
*q*3	1.086956522
Risk	79080.18441

8. Conclusions

We have presented several single period portfolio selection models in the presence of initial holdings and transaction costs. In the case the transaction costs are nonlinear, the models are nonlinear and the objective map may present several points of local minimum (maximum).

In the first three models, that is in the models of Markowitz type, the risk is measured by the variance of the return. In the last two models the risk is measured using the cumulative distribution function of the return.

Consequently the first three models are less complex than the last two models. The advantage of the last two models is that measure of risk they use is more accurate than the variance.

An efficient approach in the attempt to find a global minimum (maximum) of the objective map is computer simulation. We used an optimizer for our model based on Monte Carlo simulation. It was included in a portfolio simulation module PORTOPT which is the core of a risk management decision support system.

Extensions of our models in order to treat the discrete cases, the multiperiod cases and to include some other measures of risk is underway.

References

1. Cvitanic J., Karatzas, I. "Hedging and portfolio optimization under transaction costs: a martingale approach". *Math. Finance* 1996; 6: 133--165.
2. Davis, M.H.A., Norman A.R. "Portfolio selection with transaction costs". *Math. of Oper. Research*. 1990; 15/4: 676-713.
3. Dumas B., Luciano E. "An exact solution to a dynamic portfolio choice problem under transaction costs". *J. of Finance*. 1991; 46/2: 577-595.
4. Francis J.C., Archer S.H. *Portfolio Analysis*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1979.
5. Leland H.E. "Option pricing and replication with transaction costs". *J. of Finance.* 1985; 40/5: 1283-1301.
6. Magill M., Constantinides G. "Portfolio selection with transaction costs". *J. of Economic Theory,* 1976; 13: 245-263.
7. Markowitz H. M. "Portfolio selection", *J.of Finance*. 1952; 8: 77-91.
8. Markowitz H. M. *Portfolio Selection: Efficient Diversification of Investments*, John Wiley and Sons, New York, 1959.
9. Markowitz H. M. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, Basil Blackwell Ltd., Cambridge, Massachusetts, 1987.
10. Pogue G.A. "An extension of the Markowitz portfolio selection model to include transaction costs, short sales, leverage policies and taxes". *J. of Finance*, 1970; 25: 1005-1027.
11. Radulescu C-Z, Radulescu M., Radulescu S. " Single period selection of portfolio models for the small investor", in: *Information Technology,* The 4th International Symposium of Economic Informatics, 1999, pp. 546-552, INFOREC, Bucharest.
12. Radulescu C-Z, Radulescu M., Radulescu S. "Mathematical modelling for optimal investment decisions". *Information Society,* Proceedings of the fifth international symposium on economic informatics, 2001, pp.581-586, INFOREC, Bucharest.
13. Radulescu M., Radulescu S., Radulescu C-Z, "Computer simulation for portfolio selection models with transaction costs and initial holdings", in Proceedings of the EUROSIM 2001 Congress, "Shaping Future with Simulation", Delft, 2001, pp. 1-6.
14. Radulescu C-Z, Radulescu M., Radulescu S. "Optimal decisions for portfolio selection in the presence of initial holdings and transaction costs", Proceedings of the 4th International Workshop on Computer Science and Information Technologies CSIT'2002, Patras, Greece, 2002, pp.1-6.